Chapter 3: Complex Numbers

Ques ▶ 1 Scenario-1 : $f(x) = 3x + 1$.
Scenario-2 : $|z - 5| = 3$

[D.B., Dj.B., S.B., J.B.-18]

a. What do you mean by $\mathbb{R}$ and $\mathbb{C}$? What is the relation between them? 2

b. Show that the solution set of $2\,|f(x-2)| \le 1$ on the real line. 4

c. What does the locus of secnario-2 represent genometrically if $z = x + iy$? Sketch it. 4

Solution to the question no. 1

a The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers since all the real numbers are complex numbers with their imaginary part zero (0) So $\mathbb{R}$ is a proper subset of $\mathbb{C}$ i.e., $\mathbb{R} \subset \mathbb{C}$.

b According to the scenario-1, $f(x) = 3x + 1$

$\therefore\ f(x-2) = 3(x-2) + 1 = 3x - 6 + 1 = 3x - 5$

Now, $2|f(x-2)| \le 1$

or, $2|3x - 5| \le 1$

or, $|3x - 5| \le \frac{1}{2}$

or, $-\frac{1}{2} \le 3x - 5 \le \frac{1}{2}$ [$\because |a| \le \alpha$ implies $-\alpha \le a \le \alpha$]

or, $-\frac{1}{2} + 5 \le 3x - 5 + 5 \le \frac{1}{2} + 5$ [Adding 5 on both sides]

or, $\frac{-1+10}{2} \le 3x \le \frac{1+10}{2}$

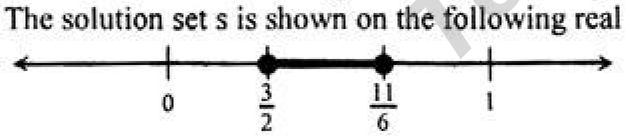
or, $\frac{9}{2} \le 3x \le \frac{11}{2}$

or, $\frac{3}{2} \le x \le \frac{11}{6}$ [Dividing both sides by 3]

$\therefore \frac{3}{2} \le x \le \frac{11}{6}$

$\therefore$ The solution set, $S = \left\{ x \in \mathbb{R} : \frac{3}{2} \le x \le \frac{11}{6} \right\}$

The solution set s is shown on the following real line:



c According to scenario-2, $|z - 5| = 3$

or, $|x + iy - 5| = 3$ [$\because z = x + iy$]

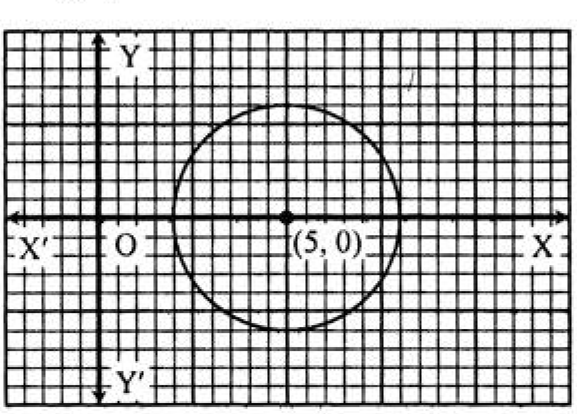
or, $|(x-5) + iy| = 3$

or, $\sqrt{(x-5)^2 + y^2} = 3$

$\therefore (x-5)^2 + (y-0)^2 = 3^2$

Which represents a circle with centre (5,0) and radius 3 units.

Now, taking 2 small squares = 1 unit in the both axes, we draw the graph of the circle.



Ques ▶ 2

Scenario-1 : $z = x + iy$ and $|z+1| + |z-1| = 4$.

Scenario-2 ; $a = p + q$, $b = p + \omega q$ and $c = p + \omega^2 q$.

[R.B., C.B., Ctg.B., B.B.-18]

a. Express $\left(\frac{1+i}{1-i}\right)^3$ in the form of A + iB. 2

b. From Scenario-1, prove that $3x^2 + 4y^2 = 12$. 4

c. From Scenario-2, show that $a^3 + b^3 + c^3 = 3(p^3 + q^3)$. 4

Solution to the question no. 2

a $\left(\frac{1+i}{1-i}\right)^3 = \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^3$

$= \left(\frac{(1+i)^2}{1-i^2}\right)^3$

$= \left(\frac{1+2i+i^2}{1-(-1)}\right)^3$ [$\because i^2 = -1$]

$= \left(\frac{1+2i-1}{2}\right)^3$

$= \left(\frac{2i}{2}\right)^3 = i^3 = -i$

$= 0 + i(-1)$

$= A + iB$ where, $A = 0$ and $B = -1$ **(Ans.)**

b From the scenario-1, $|z+1| + |z-1| = 4$

or, $|x + iy + 1| + |x + iy - 1| = 4$ [$\because z = x + iy$]

or, $|(x+1) + iy| + |(x-1) + iy| = 4$

or, $\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 4$

or, $\sqrt{(x+1)^2 + y^2} = 4 - \sqrt{(x-1)^2 + y^2}$

or, $(x+1)^2 + y^2 = 16 - 8\sqrt{(x-1)^2 + y^2} + (x-1)^2 + y^2$ [Squaring on both sides]

or, $(x+1)^2 - (x-1)^2 = 16 - 8\sqrt{(x-1)^2 + y^2}$

or, $4x = 16 - 8\sqrt{x^2 + y^2 - 2x + 1}$

or, $x = 4 - 2\sqrt{x^2 + y^2 - 2x + 1}$

or, $x - 4 = -2\sqrt{x^2 + y^2 - 2x + 1}$

or, $(x-4)^2 = 4(x^2 + y^2 - 2x + 1)$ [Squaring again]

or, $x^2 - 8x + 16 = 4x^2 + 4y^2 - 8x + 4$

$\therefore\ 3x^2 + 4y^2 = 12$ **(Proved)**

c From the scenario-2,

$a = p + q,\ b = p + \omega q,\ c = p + \omega^2 q$

$\therefore$ L.H.S $= a^3 + b^3 + c^3$

$= (p+q)^3 + (p+\omega q)^3 + (p+\omega^2 q)^3$

$= p^3 + q^3 + 3p^2q + 3pq^2 + p^3 + (\omega q)^3 + 3p^2\omega q + 3p(\omega q)^2 + p^3 + (\omega^2 q)^3 + 3p^2.\,\omega^2 q + 3p.(\omega^2 q)^2$

$= 3p^3 + q^3 + 3p^2q + 3pq^2 + \omega^3 q^3 + 3p^2q.\omega + 3p.\omega^2 q^2 + \omega^6 q^3 + 3p^2q.\omega^2 + 3p.q^2.\omega^4$

$= 3p^3 + q^3 + 3p^2q + 3pq^2 + 1.q^3 + 3p^2q.\omega + 3pq^2.\omega^2 + 1.q^3 + 3p^2.q.\omega^2 + 3pq^2.\omega$

[$\because \omega^3 = 1;\ \omega^6 = (\omega^3)^2 = 1^2 = 1;\ \omega^4 = \omega^3.\omega = 1.\omega = \omega$]

$= 3p^3 + q^3 + 3p^2q + 3pq^2 + q^3 + 3p^2q\omega + 3pq^2\omega^2 + q^3 + 3p^2q\omega^2 + 3pq^2\omega$

$= 3p^3 + 3q^3 + 3p^2q(1 + \omega + \omega^2) + 3pq^2(1 + \omega^2 + \omega)$
$= 3p^3 + 3q^3 + 3p^2q.0 + 3pq^2.0 \; [\because 1 + \omega + \omega^2 = 0]$
$= 3p^3 + 3q^3 = 3(p^3 + q^3) = $ R.H.S
$\therefore a^3 + b^3 + c^3 = 3(p^3 + q^3)$ **(Shown)**

Ques ▶3 $z_1 = 2 + 3i$, $z_2 = 1 + 2i$, $a = p\omega^2 + q + r\omega$ and $b = p\omega + q + r\omega^2$, Where ω is the complex root of cube root of unity. *[C. B.-17]*

a. Find the argument of $\frac{1}{2 - i}$. 2

b. Find the square root of $\overline{z_1 - z_2}$ by using the stem. 4

c. If $a^3 + b^3 = 0$, prove that, $2p = q + r$, $2q = r + p$ and $2r = p + q$ by using the stem. 4

Solution to the question no. 3

a Here, $\frac{1}{2 - i} = \frac{2 + i}{(2 - i)(2 + i)} = \frac{2 + i}{4 - i^2} = \frac{2 + i}{4 + 1} = \frac{2 + i}{5} = \frac{2}{5} + \frac{1}{5}i$

$\therefore$ Argument, $\theta = \tan^{-1}\left(\frac{\frac{1}{5}}{\frac{2}{5}}\right) = \tan^{-1}\left(\frac{1}{2}\right)$ **(Ans.)**

b $z_1 - z_2 = 2 + 3i - 1 - 2i = 1 + i$

$\therefore \overline{z_1 - z_2} = \overline{1 + i} = 1 - i$

Let, $\sqrt{1 - i} = x - iy$

$\Rightarrow 1 - i = x^2 - i \,.\, 2xy + i^2y^2$

$\therefore 1 - i = x^2 - y^2 - i \,.\, 2xy$

Equating the real and imaginary parts from both sides, we get,

$x^2 - y^2 = 1$ ... ... (i)

and $-2xy = -1 \;\therefore 2xy = 1$ ... ... (ii)

Now, $x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2}$
$= \sqrt{1^2 + (2xy)^2}$
$= \sqrt{1 + 1^2} = \sqrt{2}$

$\therefore x^2 + y^2 = \sqrt{2}$ ... ... (iii)

Adding (i) and (iii), we get,

$2x^2 = 1 + \sqrt{2} \Rightarrow x^2 = \frac{1}{2}(\sqrt{2} + 1)$

$\therefore x = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{2} + 1}$

Again, subtracting (i) from (iii), we get,

$2y^2 = \sqrt{2} - 1$

$\Rightarrow y^2 = \frac{1}{2}(\sqrt{2} - 1)$

$\therefore y = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{2} - 1}$

$\therefore \sqrt{1 - i} = x - iy$
$= \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{2} + 1} - i\left(\pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{2} - 1}\right)$
$= \pm\frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{2} + 1} - i\sqrt{\sqrt{2} - 1}\right)$ **(Ans.)**

c Given that, $a^3 + b^3 = 0$

$\Rightarrow (a + b)(a^2 - ab + b^2) = 0$

$\Rightarrow (a + b)(\{a^2 + (\omega + \omega^2)ab + \omega^3b^3\} = 0$

$\Rightarrow (a + b)\{a^2 + \omega ab + \omega^2 ab + \omega^3b^3\} = 0$

$\Rightarrow (a + b)\{a(a + \omega b) + \omega^2b(a + \omega b)\} = 0$

$\therefore (a + b)(a + \omega b)(a + \omega^2 b) = 0$

Either, $a + b = 0$

$\Rightarrow p\omega^2 + q + r\omega + p\omega + q + r\omega^2 = 0$

$\Rightarrow p(\omega + \omega^2) + 2q + r(\omega + \omega^2) = 0$

$\Rightarrow 2q - p - r = 0 \;\therefore 2q = r + p$ **(Proved)**

or, $a + \omega b = 0$

$\Rightarrow p\omega^2 + q + r\omega + p\omega^2 + q\omega + r\omega^3 = 0$

$\Rightarrow 2p\omega^2 + q(1 + \omega) + r(\omega + 1) = 0$

$\Rightarrow 2p\omega^2 - q\omega^2 - r\omega^2 = 0$

$\Rightarrow 2p - q - r = 0 \;\therefore 2p = q + r$ **(Proved)**

or, $a + b\omega^2 = 0$

$\Rightarrow p\omega^2 + q + r\omega + p\omega^3 + q\omega^2 + r\omega^4 = 0$

$\Rightarrow p\omega^2 + p + q + q\omega^2 + r\omega + r\omega = 0$

$\Rightarrow p(1 + \omega^2) + q(1 + \omega^2) + 2r\omega = 0$

$\Rightarrow -p\omega - q\omega + 2r\omega = 0$

$\Rightarrow 2r - p - q = 0 \;\therefore 2r = p + q$ **(Proved)**

Ques ▶4

Stem-1: $z = x + iy$; $|z + 5| + |z - 5| = 15$................ (1)

[Ctg. B.-17]

Stem-2: $\frac{2x + 3}{x - 3} < \frac{x + 3}{x - 1}$ .... (2)

a. Find the cube roots of unity. 2

b. From stem-1, find the equation of the locus. 4

c. From stem-2, solve the inequality and show them in the real lines. 4

Solution to the question no. 4

a Let, $\sqrt[3]{1} = x$ or, $x^3 = 1$ or, $x^3 - 1 = 0$

$\therefore (x - 1)(x^2 + x + 1) = 0$

Either, $x - 1 = 0$

$\therefore x = 1$

or, $x^2 + x + 1 = 0$

$\therefore x = \frac{-1 \pm \sqrt{1 - 4}}{2}$

$= \frac{1}{2}(-1 \pm i\sqrt{3})$ **(Ans.)**

b From the scenario-1, $|z + 5| + |z - 5| = 15$

or, $|x + iy + 5| + |x + iy - 5| = 15 \; [\because z = x + iy]$

or, $|x + 5 + iy| + |x - 5 + iy| = 15$

or, $\sqrt{(x + 5)^2 + y^2} + \sqrt{(x - 5)^2 + y^2} = 15$

or, $\sqrt{(x + 5)^2 + y^2} = 15 - \sqrt{(x - 5)^2 + y^2}$

or, $(x + 5)^2 + y^2 = 225 + (x - 5)^2 + y^2$
$- 30\sqrt{(x - 5)^2 + y^2}$ [squaring on both sides]

or, $(x + 5)^2 - (x - 5)^2 = 225 - 30\sqrt{x^2 - 10x + 25 + y^2}$

or, $20x = 225 - 30\sqrt{x^2 - 10x + 25 + y^2}$

or, $4x - 45 = -6\sqrt{x^2 - 10x + 25 + y^2}$

or, $16x^2 - 360x + 2025 = 36(x^2 - 10x + 25 + y^2)$

or, $16x^2 - 360x + 2025 = 36x^2 - 360x + 900 + 36y^2$

$\therefore 20x^2 + 36y^2 = 1125$; which the equation of the locus.

c Given inequality, $\frac{2x + 3}{x - 3} < \frac{x + 3}{x - 1}$

or, $\frac{2x + 3}{x - 3} - \frac{x + 3}{x - 1} < 0$

or, $\frac{(2x^2 + 3x - 2x - 3) - (x^2 - 9)}{(x - 3)(x - 1)} < 0$

or, $\dfrac{2x^2 + x - 3 - x^2 + 9}{(x - 3)(x - 1)} < 0$

or, $\dfrac{x^2 + x + 6}{(x - 3)(x - 1)} < 0$

or, $\dfrac{x^2 + 2.\frac{1}{2}.x + \left(\frac{1}{2}\right)^2 + 6 - \frac{1}{4}}{(x - 3)(x - 1)} < 0$

or, $\dfrac{\left(x + \frac{1}{2}\right)^2 + \frac{23}{4}}{(x - 3)(x - 1)} < 0$ ... ... (i)

Here, $\left(x + \frac{1}{2}\right)^2 + \frac{23}{4} > 0$

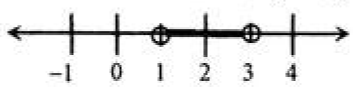
∴ The condition satisfies the inequality of (i) one sign is positive and another is negetive between $(x - 3)$ and $(x - 1)$,

Condition	Sign of $(x - 1)$	Sign of $(x - 3)$	Sign of $(x - 3)(x - 1)$
$x < 1$	–	–	+
$1 < x < 3$	+	–	–
$x > 3$	+	+	+

∴ Inequality (i) will be true if $1 < x < 3$

∴ The required solution : $1 < x < 3$ **(Ans.)**

Number line :



Ques ▶5 **Scenario-1:** $x + iy = 2e^{-i\theta}$.

Scenario-2: $F = y - 2x$.

Conditions: $x + 2y \leq 6$, $x + y \geq 4$, $x, y \geq 0$. *[J. B.-17]*

a. If $z = x + iy$ then find the locus represented by $|z + i| = |\bar{z} + 2|$. 2

b. From scenario-1, prove that $x^2 + y^2 = 4$. 4

c. Find the maximum value of F by graphical method of the linear programming in scenario-2. 4

Solution to the question no. 5

a Given that, $z = x + iy$

∴ $|z + i| = |\bar{z} + 2|$

or, $|x + iy + i| = |\overline{x + iy} + 2|$

or, $|x + iy + i| = |x - iy + 2|$

or, $|x + i(y + 1)| = |(x + 2) - iy)|$

or, $\sqrt{x^2 + (y + 1)^2} = \sqrt{(x + 2)^2 + y^2}$

or, $x^2 + (y + 1)^2 = (x + 2)^2 + y^2$

or, $x^2 + y^2 + 2y + 1 = x^2 + 4x + 4 + y^2$

or, $2y + 1 = 4x + 4$

or, $4x - 2y + 3 = 0$

Which is the required equation of the locus. **(Ans.)**

b From the scenario-1, $x + iy = 2e^{-i\theta}$

or, $x + iy = 2(\cos\theta - i \sin\theta)$

∴ $x + iy = 2\cos\theta - 2i \sin\theta$

Equating real and imaginary parts from both sides, we get,

$x = 2 \cos\theta$ and $y = - 2\sin\theta$

Now, $x^2 + y^2 = (2\cos\theta)^2 + (- 2 \sin\theta)^2$

$= 4\cos^2\theta + 4\sin^2\theta$

$= 4 (\cos^2\theta + \sin^2\theta) = 4$

∴ $x^2 + y^2 = 4$ **(Proved)**

c From the scenario-2, $F = y - 2x$

Subject to : $x + 2y \leq 6$, $x + y \geq 4$, $x, y \geq 0$

It is required to find the maximum value of F.

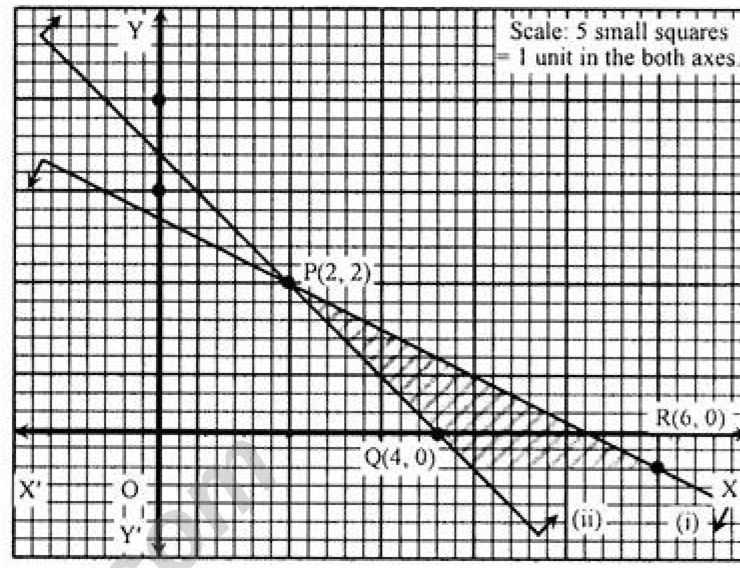
The corresponding equations of the constraint inequalities are.

$x + 2y = 6 \Rightarrow \frac{x}{6} + \frac{y}{3} = 1$ ... ... (1)

$x + y = 4 \Rightarrow \frac{x}{4} + \frac{y}{4} = 1$ ... ... (2)

$x = 0$ and $y = 0$ ... ... (3)

Now, the draw the graphs of the constraint inequalities with the help of the system of equations (1) to (3) and find the feasible region of the system of constraints.



The shaded region of the graph is the feasible region determined by the constraint inequalities of the problem.

We use corner Point Method to determine the maximum value of F. From the graph, it is found that the corner points of the feasible region are P(2, 2), Q (4,0) and R (6, 0). Now, we evaluate F at each corner point as follows :

Corner point	$F = y - 2x$
P (2, 2)	$F = 2 - 2 \times 2 = - 2$
Q (4, 0)	$F = 0 - 2 \times 4 = - 8$
R (6, 0)	$F = 0 - 2 \times 6 = - 12$

From the above table, we get the maximum value of F is –2 at the point (2, 2) **(Ans.)**

Ques ▶6 $f(x) = |x - 3|$

$g(x) = p + qx + rx^2$ *[B. B.-17]*

a. Find the square root of $15 + 8i$. 2

b. if $f(x) < \frac{1}{7}$ then prove that, $|x^2 - 9| < \frac{43}{49}$. 4

c. If $p + q + r = 0$, then prove that, $\{g(\omega)\}^3 + \{g(\omega^2)\}^3 = a^x pqr$, Where ω is a complex cube roots of unity and $a = x = 3$. 4

Solution to the question no. 6

a Let, $p = 15 + 8i = 16 + 8i - 1$

$= 4^2 + 2.\ 4.\ i + i^2 = (4 + i)^2$

∴ $\sqrt{p} = \pm (4 + i)$ **(Ans.)**

b Given that,

$f(x) = |x - 3|$ and $f(x) < \frac{1}{7}$

∴ $|x - 3| < \frac{1}{7} \Rightarrow -\frac{1}{7} < x - 3 < \frac{1}{7}$

$\Rightarrow -\frac{1}{7} + 3 < x - 3 + 3 < \frac{1}{7} + 3$

$\Rightarrow \frac{-1 + 21}{7} < x < \frac{1 + 21}{7}$

$\Rightarrow \frac{20}{7} < x < \frac{22}{7}$

$$\Rightarrow \frac{400}{49} < x^2 < \frac{484}{49} \quad \text{[Squaring on both sides]}$$

$$\Rightarrow \frac{400}{49} - 9 < x^2 - 9 < \frac{484}{49} - 9$$

[Subtracting 9 from both sides]

$$\Rightarrow \frac{400-441}{49} < x^2 - 9 < \frac{484-441}{49}$$

$$\Rightarrow \frac{-41}{49} < x^2 - 9 < \frac{43}{49}$$

$$\Rightarrow \frac{-43}{49} < \frac{-41}{49} < x^2 - 9 < \frac{43}{49}$$

$$\Rightarrow \frac{-43}{49} < x^2 - 9 < \frac{43}{49}$$

$\therefore |x^2 - 9| < \frac{43}{49}$ **(Proved)**

c Given that, $g(x) = p + qx + rx^2$

$\therefore g(\omega) = p + q\omega + r\omega^2$

and $g(\omega^2) = p + q\omega^2 + r\omega^4 = p + q\omega^2 + r\omega$

$\therefore \{g(\omega)\}^3 + \{g(\omega^2)\}^3 = (p + q\omega + r\omega^2)^3 + (p + q\omega^2 + r\omega)^3$

Let, $p + q\omega + r\omega^2 = x$ and $p + q\omega^2 + r\omega = y$

$\therefore$ L.H.S. $= x^3 + y^3$

$= (x + y)(x^2 - xy + y^2)$

$= (x + y)\{x^2 + (-1)xy + y^2\}$

$= (x + y)\{x^2 + (\omega^2 + \omega)xy + y^2\}$ $[\because \omega^2 + \omega + 1 = 0]$

$= (x + y)(x^2 + \omega^2xy + \omega xy + y^2)$

$= (x + y)(x^2 + \omega xy + \omega^2xy + y^2\omega^3)$

$= (x + y)\{x(x + \omega y) + \omega^2y(x + \omega y)\}$ $[\because \omega^3 = 1]$

$= (x + y)(x + \omega y)(x + \omega^2y)$

Now, $x + y = p + q\omega + r\omega^2 + p + q\omega^2 + r\omega$

$= 2p + q(\omega + \omega^2) + r(\omega^2 + \omega) = 2p - q - r$

$x + \omega y = p + q\omega + r\omega^2 + p\omega + q\omega^3 + r\omega^2$

$= p(1 + \omega) + q(\omega + 1) + 2r\omega^2$

$= p(-\omega^2) + q(-\omega^2) + 2r\omega^2 = \omega^2(2r - p - q)$

$x + \omega^2y = p + q\omega + r\omega^2 + p\omega^2 + q\omega^4 + r\omega^3$

$= p + q\omega + r\omega^2 + p\omega^2 + q\omega + r$

$= p(1 + \omega^2) + 2q\omega + r(1 + \omega^2)$

$= p(-\omega) + 2q\omega + r(-\omega) = \omega(2q - p - r)$

$\therefore (x + y)(x + \omega y)(x + \omega^2y)$

$= (2p - q - r)\omega^2(2r - p - q)\omega(2q - p - r)$

$= \omega^3(2p - q - r)(2r - p - q)(2q - p - r)$

$= \{2p - (q + r)\}\{2r - (p + q)\}\{2q - (p + r)\}$

$= \{2p - (-p)\}\{2r - (-r)\}\{2q - (-q)\}$

$[\because p + q + r = 0]$

$= 3p.3r.3q = 27pqr = 3^3 pqr$

$= a^xpqr$ $[\because a = x = 3]$ = R.H.S

$\therefore \{g(\omega)\}^3 + \{g(\omega^2)\}^3 = a^xpqr$ **(Proved)**

Ques ▶ 7 $z = px + qy$;

Constraints: $x + y \le 7$, $2x + 5y \le 20$, $x \ge 0$, $y \ge 0$

[Mirzapur Cadet College, Tangail]

a. Find the square root of 2i. 2

b. If p = 3, q = 4 find the maximum value of z using graph. 4

c. Show that $4(x^2 - y^2) = \frac{a}{x} + \frac{b}{y}$, if $\sqrt[3]{a + ib} = z$ when p = 1 and q = i. 4

Solution to the question no. 7

a Let, $x = \sqrt{2i}$

$= \sqrt{1 + 2i - 1}$

$= \sqrt{1^2 + 2.1.i + i^2}$

$= \sqrt{(1 + i)^2}$

$\therefore$ $x = \pm(1 + i)$ **(Ans.)**

b Given that, $z = px + qy$

If p = 3, q = 4 then, $z = 3x + 4y$

$\therefore$ The objective function, $Z_{max} = 3x + 4y$

Subject to : $x + y \le 7$, $2x + 5y \le 20$, $x, y \ge 0$

The corresponding equations of the constraint inequalities are :

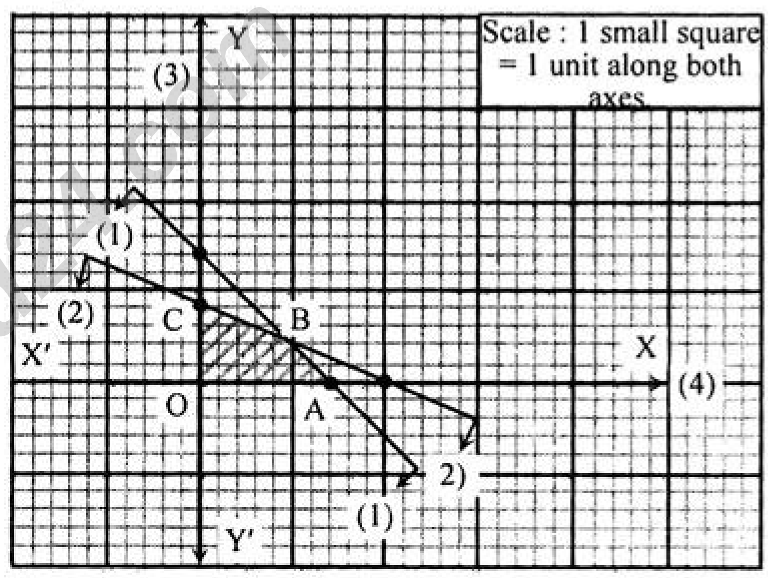
$x + y = 7$ or, $\frac{x}{7} + \frac{y}{7} = 1$ ……… (1)

$2x + 5y = 20$ or, $\frac{x}{10} + \frac{y}{4} = 1$ ……… (2)

$x = 0$ ……… (3)

$y = 0$ ……… (4)

Now, we draw the graph of the equations (1) to (4) and determine the feasible region formed by the constraint inequalities.



In the figure, the shaded area indicates the feasible region whose corner points are–

O(0, 0), A(7, 0), B(5, 2), and C(0, 4)

Now at O(0, 0), $Z = 3 \times 0 + 4 \times 0 = 0$

At A(7, 0), $Z = 3 \times 7 + 4 \times 0 = 21$

At B(5, 2), $Z = 3 \times 5 + 4 \times 2 = 23$

At C(0, 4), $Z = 3 \times 0 + 4 \times 4 = 16$

Clearly, the maximum value of Z occurs at B(5, 2).

$\therefore$ The maximum value, $Z_{max} = 23$ **(Ans.)**

c Given that, $z = px + qy$

If p = 1 and q = i then, $z = x + iy$

According to question $\sqrt[3]{a + ib} = x + iy$

Taking cube $a + ib = (x + iy)^3$ [Taking cube]

or, $a + ib = x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3$

$= x^3 + 3x^2yi - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + 3x^2yi - iy^3$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$

Equating real and imaginary parts, we get,

$a = x^3 - 3xy^2$ and $b = 3x^2y - y^3$

Now, R.H.S $= \frac{a}{x} + \frac{b}{y}$

$$= \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y}$$
$$= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y}$$
$$= x^2 - 3y^2 + 3x^2 - y^2$$
$$= 4(x^2 - y^2)$$
$$= \text{L.H.S}$$
$\therefore\ 4(x^2 - y^2) = \frac{a}{x} + \frac{b}{y}$ **(Shown)**

Ques ▶ 8 $z_1 = a + ib$ and $z_2 = c + id$, where a, b, c and d are real numbers. *[Mymensingh Girls' Cadet College, Mymensingh]*

a. Why is every real number it self a complex number? 2

b. If $a = -8$ and $b = -6$, find the square root of z_1. 4

c. If $x:y = z_1 : z_2$, show that $(c^2 + d^2)x^2 - 2(ac + bd)xy + (a^2 + b^2)y^2 = 0$. 4

Solution to the question no. 8

a A number of the form a + ib is called complex number, where $i = \sqrt{-1}$ and a, b are any real numbers.

In a complex number a + ib, a is called real part and b is called imaginary part of the number.

If b = 0, then a + ib = a is called purely real number. Therefore, every neal number is a complex number with its imaginary part zero.

b Given that, $a = -8$ and $b = -6$

$\therefore\ z_1 = -8 + (-6)i$
$= -8 - 6i$

Now, $-8 - 6i = 1 - 6i - 9$
$= 1 - 2.3i + (3i)^2$
$= (1 - 3i)^2$

$\therefore$ The square root of $-8 - 6i = \pm\sqrt{(1 - 3i)^2} = \pm(1 - 3i)$ **(Ans.)**

c Given that, $x : y = z_1 : z_2$

or, $\frac{x}{y} = \frac{z_1}{z_2}$

or, $\frac{x}{y} = \frac{a + ib}{c + id}$

or, $x(c + id) = y(a + ib)$

or, $cx + idx = ay + iby$

or, $idx - iby = ay - cx$

or, $i(dx - by) = (ay - cx)$

or, $i^2(dx - by)^2 = (ay - cx)^2$ [Squaring or both sides]

or, $-(d^2x^2 - 2bdxy + b^2y^2) = a^2y^2 - 2acxy + c^2x^2$

or, $-d^2x^2 + 2bdxy - b^2y^2 = a^2y^2 - 2acxy + c^2x^2$

or, $(a^2 + b^2)y^2 + (c^2 + d^2)x^2 = 2(ac + bd)xy$

$\therefore\ (c^2 + d^2)x^2 - 2(ac + bd)xy + (a^2 + b^2)y^2 = 0$ **(Shown)**

Ques ▶ 9 Suppose $p = \frac{-1 + \sqrt{-3}}{2}$, $q = \frac{-1 - \sqrt{-3}}{2}$ and

$f(x) = \frac{1 - ix}{1 + ix} - (a - ib)$. *[Rajshahi Cadet College, Rajshahi]*

a. Find the square root of $(2 - 7i)$ 2

b. Show that $p^n + q^n = 2$ or -1, where it is divisible by 3 or n is any other natural number respectively. 4

c. If a, b, ∈ R and $a^2 + b^2 - 1 = 0$, then show that the equation $f(x) = 0$ has a real root. 4

Solution to the question no.9

a Square root of $(2 - 7i) = \sqrt{2 - 7i}$

Let, $\sqrt{2 - 7i} = a + ib$; where, a and b are real number ....(i)

Or, $2 - 7i = (a + ib)^2$

Or, $2 - 7i = a^2 + 2ab\,i + i^2b^2$

Or, $2 - 7i = (a^2 - b^2) + 2abi$ [$\because i^2 = -1$]

Equating the real and imaginary number from both sides,

$a^2 - b^2 = 2$........(ii)

$2ab = -7$..........(iii)

Now, $a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2}$
$= \sqrt{(2)^2 + (-7)^2}$
$= \sqrt{4 + 49}$
$= \sqrt{53}$.

$\therefore\ a^2 + b^2 = \sqrt{53}$ – – – (iv)

By adding equation (i) and (iv) we get,

$a^2 - b^2 + a^2 + b^2 = 2 + \sqrt{53}$

Or, $2a^2 = 2 + \sqrt{53}$.

Or, $a^2 = \frac{1}{2}(2 + \sqrt{53})$.

Or, $a = \sqrt{1 + \frac{\sqrt{53}}{2}}$

By subtracting equation (i) from (iv) we get,

$a^2 + b^2 - a^2 + b^2 = \sqrt{53} - 2$.

Or, $2b^2 = \sqrt{53} - 2$

Or, $b^2 = \frac{\sqrt{53}}{2} - 1$

Or, $b = \sqrt{\frac{\sqrt{53}}{2} - 1}$

Putting the value of a, b in equation (i) we get,

$\sqrt{2 - 7i} = \sqrt{1 + \frac{\sqrt{53}}{2}} + i\sqrt{\frac{\sqrt{53}}{2} - 1}$. **(Ans.)**

b Similar to Question No: 22(c), Chapter-7.

c Similar to Question No: 14(c).

Ques ▶ 10 Imaginary number and complex number—
[Joypurhat Girl's Cadet College]

a. What is the difference of imaginary number and complex number? 2

b. Prove that, the product of two odd neutral number is always odd. 4

c. If $a^2 + b^2 = 1$, then show that a real value of 'x' and 'y' will satisfy the equation $\frac{x - iy}{x + iy} = a - ib$, where a, b ∈ R.

Solution to the question no. 10

a The imaginary numbers are the positive square root negative numbers that can be written as a real number multiplied by the imaginary unit $i = \sqrt{-1}$.

On the other hand, the complex numbers are numbers that can be written as $z = a + ib$, where $a, b \in \mathbb{R}$. The number a is called the real part of z and b is called the imaginary part of z.

If the real part $a = 0$, then $z = ib$ is a pure imaginary complex number. Thus, every imaginary number is a complex number with real part zero, but each complex number is not pure imaginary complex.

b Let, m and n be two odd natural numbers,

Then, by the definition of odd number,

$m = 2a + 1$ and $n = 2b + 1$, where a, b are non-negative integers.

Now, $mn = (2a + 1)(2b + 1)$

$= 4ab + 2a + 2b + 1$

$= 2(2ab + a + b) + 1$

$= 2k + 1$

Where $k = (2ab + a + b)$ is a non-negative integer.

$\therefore$ mn is an odd natural number.

Therefore, the product of two odd natural numbers is always odd. **(Proved)**

c Similar to the Question No: 14(c).

Ques ▶ 11 $Z_1 = a + ib$, $Z_2 = c + id$ are two complex numbers and $p + q = -r$. *[Pabna Cadet College, Pabna]*

a. Express $\frac{3 + i4}{3 - i4}$ in the form of A + iB. 2

b. If $z_1 z_2 = x:y$ then show that $(c^2 + d^2)x^2 - 2(ac + bd)xy + (a^2 + b^2)y^2 = 0$. 4

c. Prove $(p + q\omega + r\omega^2)^3 + (p + q\omega^2 + r\omega)^3 = 27pqr$ where ω is a cube root of unity. 4

Solution to the question no. 11

a $\frac{3 + i4}{3 - i4} = \frac{(3 + 4i)(3 + 4i)}{(3 - 4i)(3 + 4i)}$

$= \frac{(3 + 4i)^2}{(3)^2 - (4i)^2}$

$= \frac{(3)^2 + 2.3.4i + (4i)^2}{9 + 16}$

$= \frac{9 + 24i - 16}{25}$

$= \frac{-7}{25} + i\frac{24}{25}$ which is of the form A + iB **(Ans.)**

b See the Question No: 8(c).

c Given that, $p + q = -r$

Let, $p + q\omega + r\omega^2 = x$ and $p + q\omega^2 + r\omega = y$

$\therefore \quad x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$= (x + y)\{x^2 + (\omega + \omega^2)xy + \omega^3 y^2\}$

$[\therefore 1 + \omega + \omega^2 = 0]$

$= (x + y)(x + \omega y)(x + \omega^2 y)$

Now, $x + y = 2p + q(\omega + \omega^2) + r(\omega + \omega^2)$

$= 2p - q - r$

$x + \omega y = p + q\omega + r\omega^2 + p\omega + q\omega^3 + r\omega^2$

$= p(1 + \omega) + q(1 + \omega) + 2r\omega^2$

$= -p\omega^2 - q\omega^2 + 2r\omega^2$

$= (-p - q + 2r)\omega^2$

$x + \omega^2 y = p + q\omega + r\omega^2 + p\omega^2 + q\omega^4 + r\omega^3$

$= p(1 + \omega^2) + q(\omega + \omega) + r(1 + \omega^2)$

$= -p\omega + 2q\omega - r\omega$

$= (-p + 2q - r)\omega$

$\therefore \quad (p + q\omega + r\omega^2)^3 + (p + q\omega^2 + r\omega)^3$

$= (2q - p - r)(2p - q - r)(2r - p - q)\omega^3$

$= (2q - p - r)(2p - q - r)(2r - p - q)$

$= \{3q - (p + q + r)\}\{3p - (p + q + r)\}\{3r - (p + q + r)\}$

$(3q - 0)(3p - 0)(3r - 0)$ $[\because p + q + r = 0]$

$= 3q.3p.3r$

$= 27pqr$

$\therefore \quad (p + q\omega + r\omega^2)^3 + (p + q\omega^2 + r\omega)^3 = 27pqr$ **(Proved)**

Ques ▶ 12 $z = 3x + 4y$

Constraints : $x \le 2y + 2$, $x \ge 6 - 2y$, $y \le x$, $x \le 6$.

[Rangpur Cadet College, Rangpur]

a. If $y = 1$ and $|z| < 1$, find the limits of x. 2

b. Find the minimum value of the objective function z under the given constraints. 4

c. If $x = 1$, $y = \sqrt{-1}$ and $\frac{z}{\bar{z}}$ A + iB, find argument of A – iB 4

Solution to the question no. 12

a Given that, $y = 1$ and $|z| < 1$

$\therefore \quad |3x + 4y| < 1$

or, $|3x + 4| < 1$

or, $-1 < 3x + 4 < 1$

or, $-1 - 4 < 3x < 1 - 4$

or, $-5 < 3x < -3$

or, $\frac{-5}{3} < \frac{3x}{3} < -\frac{3}{3}$

$\therefore \quad -\frac{5}{3} < x < -1$ **(Ans.)**

b Given that, object function, $z = 3x + 4y$

Subject to, $x \le 2y + 2$, $x \ge 6 - 2y$, $y \le x$, $x \le 6$

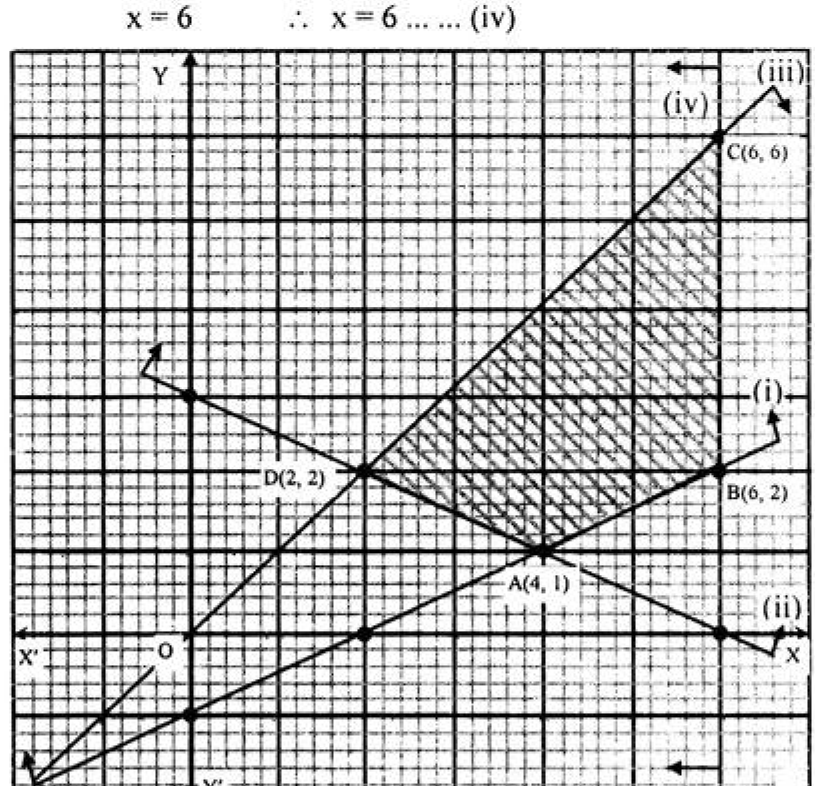
We draw the graph of the corresponding equations of the constraint inequalities and determine the feasible region. then we get,

$\therefore \quad x = 2y + 2 \Rightarrow \frac{x}{2} + \frac{y}{-1} = 1$ ... ... (i)

$x = 6 - 2y \Rightarrow \frac{x}{6} + \frac{y}{3} = 1$ ... ... (ii)

$y = x \Rightarrow y = x$ ... ... (iii)

$x = 6 \quad \therefore \ x = 6$ ... ... (iv)



In the graph, the shaded area indicates the feasible region and the corner points of the feasible region are : A(4, 1), B(6, 2), C(6, 6), D(2, 2)

At the point A(4, 1), z = 16

At the point B(6, 2), z = 26

At the point C(6, 6), z = 42

At the point D(2, 2), z = 14

Clearly, z attains its minimum value at D(2, 2) and the minimum value, $Z_{min} = 14$

c Given that, $x = 1, y = \sqrt{-1} = i$

and $z = 3x + 4y$

$\therefore \quad z = 3 + 4i$

$\bar{z} = 3 - 4i$

Now, $\frac{z}{\bar{z}} = \frac{3+4i}{3-4i} = \frac{(3+4i)^2}{(3-4i)(3+4i)}$

$= \frac{9 + 16i^2 + 2.3.4i}{3^2 - (4i)^2}$

$= \frac{9 - 16 + 24i}{9 + 16}$

$= \frac{-7 + 24i}{25}$

$= -\frac{7}{25} + \frac{24}{25}i$

According to the question, $\frac{z}{\bar{z}} = A + iB = -\frac{7}{25} + \frac{24}{25}i$

$\therefore \quad A - iB = -\frac{7}{25} - \frac{24}{25}i$

Now, $\theta = \tan^{-1}\left|\frac{-\frac{24}{25}}{-\frac{7}{25}}\right|$

$= \tan^{-1}\left(\frac{24}{7}\right)$

Hence, the point is located on the third quarter.

$\therefore$ The required argument $= \tan^{-1}\left(\frac{24}{7}\right) - \pi$ **(Ans.)**

Ques ▶ 13 $z = -1 + i\sqrt{3}$ and $z_1 = a + ib$

[Cumilla Cadet College, Cumilla]

a. For all p, q ∈ ℝ, prove that $|p + q| \le |p| + |q|$ 2

b. Prove that, $(z)^4 + \left(\overrightarrow{z}\right)^4 = -16$ 4

c. If $\sqrt[3]{z_1} = x + iy$ then show that, $4(x^2 - y^2) = \frac{a}{x} + \frac{b}{y}$. 4

Solution to the question no. 13

a $(|p| + |q|)^2$

$= |p|^2 + 2|p||q| + |q|^2$

$= p^2 + 2|pq| + q^2$ $[\because |p|^2 = p^2, |q|^2 = q^2, |p||q| = |pq|]$

or, $(|p| + |q|)^2 \ge p^2 + 2pq + q^2$ $[\because |pq| \ge pq]$

or, $(|p| + |q|)^2 \ge (p + q)^2$

or, $(|p| + |q|)^2 \geqq (|p + q|)^2$

or, $|p + q|^2 \le (|p| + |q|)^2$

$\therefore \quad |p + q| \le |p| + |q|$ **(Proved)**

b Given that, $z = -1 + i\sqrt{3}$

$= -1 + \sqrt{-3}$ $[\therefore i^2 = -1]$

$\therefore \quad \bar{z} = -1 - \sqrt{-3}$

We know that,

$\omega = \frac{-1 + \sqrt{-3}}{2}$

and, $\omega^2 = \frac{-1 - \sqrt{-3}}{2}$

So that, $2\omega = -1 + \sqrt{-3}$ and $2\omega^2 = -1 - \sqrt{-3}$

L.H.S. $= (z)^4 + (\bar{z})^4$

$= (-1 + \sqrt{-3})^4 + (-1 - \sqrt{-3})^4$

$= (2\omega)^4 + (2\omega^2)^4$

$= 2^4(\omega^4 + \omega^8)$

$= 16(\omega^3.\omega + \omega^6.\omega^2)$

$= 16(\omega + \omega^2)$ $[\because \omega^3 = 1, \omega^6 = 1]$

$= 16(-1)$ $[\because 1 + \omega + \omega^2 = 0]$

$= -16$

= R.H.S **(Proved)**

c Given that, $\sqrt[3]{z_1} = \sqrt[3]{a + ib} = x + iy$

So that, $a + ib = (x + iy)^3$

or, $a + ib = x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3$

$= x^3 + 3x^2yi - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + 3x^2yi - iy^3$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$

Now, real and imaginary part we get,

$a = x^3 - 3xy^2$ and $b = 3x^2y - y^3$

Now, R.H.S $= \frac{a}{x} + \frac{b}{y}$

$= \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y}$

$= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y}$

$= x^2 - 3y^2 + 3x^2 - y^2$

$= 4(x^2 - y^2)$

= L.H.S

$\therefore \quad 4(x^2 - y^2) = \frac{a}{x} + \frac{b}{y}$ **(Shown)**

Ques ▶ 14 **Scenario:** $Z_1 = a + ib$, $Z_2 = c + ib$; a, b ∈ R

[Faujdarhat Cadet College, Chattogram]

a. If ω is a cube root of unity, Show that $(1 - \omega + \omega^2)^2 + (1 + \omega - \omega^2)^2 = -4$ 2

b. According to scenario – if $x : y = z_1 : z_2$ then show that $(c^2 + d^2)x^2 - 2(ac + bc)xy + (a^2 + b^2)y^2 = 0$ 4

c. If $a^2 + b^2 = 1$, Prove that a real value of x will satisfy the equation $\frac{1 + (i)3x}{1 + ix} = z_2$ 4

Solution to the question no. 14

a L.H.S $= (1 - \omega + \omega^2)^2 + (1 + \omega - \omega^2)^2$

$= (1 + \omega + \omega^2 - 2\omega)^2 + (1 + \omega + \omega^2 - 2\omega)^2$

$= (-2\omega)^2 + (-2\omega^2)^2$ $[\because 1 + \omega + \omega^2 = 0]$

$= 4\omega^2 + 4\omega^4$

$= 4\omega^2 + 4\omega^3.\omega$

$= 4(\omega^2 + \omega)$ $[\because \omega^3 = 1]$

$= 4(-1) \quad [\because \omega + \omega^2 + 1 = 0]$

$= -4$

= R.H.S **(Shown)**

b See the Question No: 8(c).

c Given that, $c^2 + d^2 = 1$

and $\frac{1 + (i)^3 x}{1 + ix} = z_2$

or, $\frac{1 - ix}{1 + ix} = c + id \quad [\because i^2 = -1]$

or, $\frac{1 + ix}{1 - ix} = \frac{1}{c + id}$ [Invertendo]

or, $\frac{1 + ix + 1 - ix}{1 + ix - 1 + ix} = \frac{1 + c + id}{1 - c - id}$ [Componendo-Dividendo]

or, $\frac{2}{2ix} = \frac{1 + c + id}{1 - c - id}$

or, $ix = \frac{1 - c - id}{1 + c + id}$

$= \frac{(1 - c - id)(1 + c - id)}{(1 + c + id)(1 + c - id)}$

$= \frac{(1 - id)^2 - c^2}{(1 + c)^2 + d^2}$

$= \frac{1 - 2id + (id)^2 - c^2}{1^2 + 2c + c^2 + d^2}$

$= \frac{1 - 2id - (c^2 + d^2)}{1 + 2c + c^2 + d^2}$

$= \frac{1 - 2id - 1}{2 + 2c}$

$= \frac{-2id}{2(1 + c)} = -\frac{id}{1 + c}$

$\therefore \quad ix = i \cdot \frac{-d}{1 + c}$

$\therefore \quad x = \frac{-d}{1 + c}$, which is real.

Therefore, a real value of x satiesfies the given equation. **(Proved)**

Ques ▶ 15 Scenario-1: $a = \sqrt[4]{-144}$

Scenario-2: $z_1 = 1 + i\sqrt{3}$ and $z_2 = \sqrt{3} - i$

[Jhenidah Cadet College, Jhenidah]

a. If the complex roots of unity are ω, ω^2 then find the value of $(-1 + \sqrt{-3})^7 + (-1 - \sqrt{-3})^7$. 2

b. From scenario-1: Find the value of a. 4

c. From scenario-2: Show that $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. 4

Solution to the question no. 15

a Here, $(-1 + \sqrt{-3})^7 + (-1 - \sqrt{-3})^7$

$= \left\{\left(\frac{-1 + \sqrt{-3}}{2}\right)^7 + \left(\frac{-1 - \sqrt{-3}}{2}\right)^7\right\} . 2^7$

$= \left(\frac{-1 + i\sqrt{3}}{2}\right)^7 + \left(\frac{-1 - i\sqrt{3}}{2}\right)^7 . 2^7$

[∵ complex cube roots of unity]

$= \{\omega^7 + (\omega^2)^7\} . 2^7$

$= 2^7(\omega + \omega^2)$

$= 128.(-1)$

$= -128$ **(Ans.)**

b Given that, $a = \sqrt[4]{-144}$

Now, $\sqrt[4]{-144} = (-1.\ 144)^{\frac{1}{4}} = (i^2.\ 12^2)^{\frac{1}{4}}$

$= \{6(\pm 2i\}^{\frac{1}{2}} = \{6(\pm 2i)\}^{\frac{1}{2}}$

$= \{6(1 \pm 2i - 1)\}^{\frac{1}{2}} = \{6(1 \pm 2i + i^2)\}^{\frac{1}{2}}$

$= \{6(1 \pm i)^2\}^{\frac{1}{2}} = \pm\sqrt{6}(1 \pm i)$

$\therefore \quad a = \pm\sqrt{6}(1 \pm i)$ **(Ans.)**

c Given that, $= z_1 = 1 + i\sqrt{3}$

$z_2 = \sqrt{3} - i$

$\therefore \quad \frac{z_1}{z_2} = \frac{1 + i\sqrt{3}}{\sqrt{3} - i} = \frac{(1 + i\sqrt{3})(\sqrt{3} + i)}{(\sqrt{3} - i)(\sqrt{3} + i)}$

$= \frac{\sqrt{3} + 3i + i - \sqrt{3}}{3 + 1}$

$= \frac{4i}{4} = i$

L.H.S. $= \arg\left(\frac{z_1}{z_2}\right) = \tan^{-1}\frac{1}{0} = \tan^{-1}\tan\frac{\pi}{2} = \frac{\pi}{2}$

R.H.S. $= \arg(z_1) - \arg(z_2)$

$= \arg(1 + i\sqrt{3}) - \arg(\sqrt{3} - i)$

$= \tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \tan^{-1}\left(-\tan\frac{\pi}{6}\right)$

$= \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{2}$

$\therefore \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ **(Shown)**

Ques ▶ 16 Scenario-1 : A man eat two types of food X and Y. The quantities of three types of nutrition N_1, N_2, N_3. Prices of food and minimum daily necessity of nutrition are as follows :

Name	X	Y	Daily minimum
	40.00	50.00	necessity
N_1	600	240	1200
N_2	300	300	1200
N_3	120	360	720

Scenerio-2: $u, v, w \in \mathbb{R}$, $w \neq 0$.

[Ideal School & College, Motijheel, Dhaka]

a. Find the modulus and argument of $1 - i$. 2

b. If $uw = vw$, then prove from scenerio-2 that $u = v$. 4

c. From Scenerio-1. Find out a food combination with the help of linear programming that can express the required nutrition of that man at a minimum cost. 4

Solution to the question no. 16

a Let, $z = 1 - i$

$\therefore \quad |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

and $\theta = \tan^{-1}\left|\frac{-1}{1}\right| = \tan^{-1}|-1| = \frac{\pi}{4}$

Here, the point $(1, -1)$ lies in the 4th quadrant.

$\therefore \quad$ Argument $= -\theta = -\frac{\pi}{4}$

$\therefore \quad$ The modulus $= \sqrt{2}$ and argument $= -\frac{\pi}{4}$ **(Ans.)**

b Given that, $u, v, w \in \mathbb{R}$ and $w \neq 0$

$\therefore$ There exists $w^{-1} \in \mathbb{R}$ such that $ww^{-1} = w^{-1}w = 1$

Now, $uw = vw$

or, $(uw)w^{-1} = (vw)w^{-1}$ [Uniqueness of multiplication]

or, $u(ww^{-1}) = v(ww^{-1})$ [Associative law]

or, $u.1 = v.1$ [Multiplicative invers]

$\therefore$ $u = v$ [Multiplicative identity] **(Proved)**

c Let, the number of units of x food = x and the number of units of Y food = y.

The objective is to fulfill the daily minimum requirement of nutrition at a least cost.

So, the objective function, $Z_{min} = 40x + 50y$

Subject to : $600x + 240y \geq 1200$

$300x + 300y \geq 1200$

$120x + 360y \geq 720$

$x \geq 0, y \geq 0$

The corresponding equations of the constraint inequalities are :

$600x + 240y = 1200$

$\Rightarrow \frac{x}{2} + \frac{y}{5} = 1$ ……… (i)

$300x + 300y = 1200$

$\Rightarrow \frac{x}{4} + \frac{y}{4} = 1$ ……… (ii)

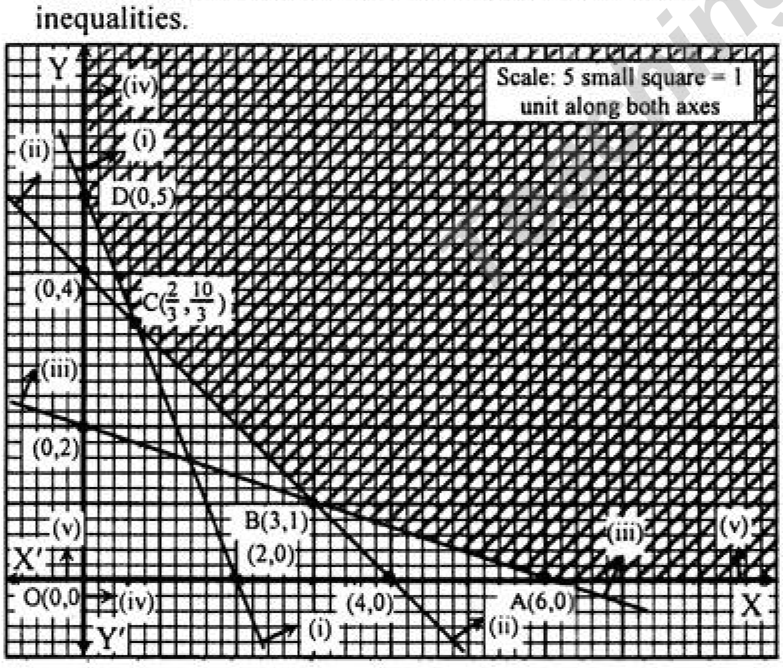
$120x + 360y = 720$

$\Rightarrow \frac{x}{6} + \frac{y}{2} = 1$ ……… (iii)

$x = 0$ ……… (iv)

$y = 0$ ……… (v)

Now, we draw the graph of the equations (i) to (v) and determine the feasible ragion formed by the constraint inequalities.



In the figure, the shaded area indicates the feasible region whose corner point are A(6, 0), B(3, 1), $C\left(\frac{2}{3}, \frac{10}{3}\right)$ and D(0, 5)

Now, at A(6, 0), $Z = 40 \times 6 + 0 \times 50 = 240$

At B(3, 1), $Z = 40 \times 3 + 50 \times 1 = 170$

At $C\left(\frac{2}{3}, \frac{10}{3}\right)$, $Z = 40 \times \frac{2}{3} + 50 \times \frac{10}{3} = \frac{580}{3}$

At D(0, 5), $Z = 40 \times 0 + 50 \times 5 = 250$

Clearly, Z attains its minimum value at B(3, 1). The food combination will be consisted with 3 units of X food, 1 unit of Y food and the minimum cost will be 170. **(Ans.)**

Ques ▶ 17 (i) $\overline{Z} = a + ib$ (ii) $\sqrt[3]{x - iy} = a - ib$

[Viqarunnisa Noon School & College, Dhaka]

a. Find the Square roots of $-7i$ — 2

b. (i) In case of (i), if $z\overline{z} - 3iz = 7 + 9i$, find z. — 4

c. In case of (ii) prove that, $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$ — 4

Solution to the question no. 17

a Square root of $-7i = \pm\sqrt{-7i}$

$= \pm\sqrt{7 \times (-i)}$

$= \pm\frac{\sqrt{7}}{\sqrt{2}}\sqrt{-2i}$

$= \pm\frac{\sqrt{7}}{\sqrt{2}}\sqrt{1^2 - 2.1(i) + i^2}$

$= \pm\frac{\sqrt{7}}{\sqrt{2}}\sqrt{(1 - i)^2}$

$= \pm\frac{\sqrt{7}}{\sqrt{2}}(1 - i)$ **(Ans.)**

b Given that,

$\overline{Z} = a + ib$

$\therefore$ $Z = a - ib$

and $Z\overline{Z} - 3iz = 7 + 9i$

or, $(a + ib)(a - ib) - 3iz = 7 + 9i$

or, $a^2 - i^2b^2 - 3iz = 7 + 9i$

or, $(a^2 + b^2) - 3zi = 7 + 9i$

Equating real and imaginary parts from both sides, we get,

$a^2 + b^2 = 7$

$-3z = 9$

$\therefore$ $z = -3$ **(Ans.)**

c Given that,

$\sqrt[3]{x - iy} = a - ib$

or, $(x - iy)^{\frac{1}{3}} = a - ib$

or, $\left\{(x - iy)^{\frac{1}{3}}\right\}^3 = (a - ib)^3$

or, $x - iy = a^3 - 3a^2ib + 3ai^2b^2 - i^3b^3$

$= a^3 - 3a^2ib - 3ab^2 + ib^3$

$x - iy = (a^3 - 3ab^2) + (b^3 - 3a^2b)i$

Equating real and imaginary parts from both sides,

$x = a^3 - 3ab^2$

$y = -(b^3 - 3a^2b) = 3a^2b - b^3$

Now, L.H.S. $= \frac{x}{a} - \frac{y}{b}$

$= \frac{a^3 - 3ab^2}{a} - \frac{3a^2b - b^3}{b}$

$= a^2 - 3b^2 - 3a^2 + b^2$

$= -2a^2 - 2b^2$

$= -2(a^2 + b^2) =$ R.H.S.

$\therefore$ $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$ **(Proved)**

Ques ▶ 18 Scenario-1: a, b $\in$ R and i is an imaginary number

Scenario-2 : $3y - x \le 10$, $x + y \le 6$, $x - y \le 2$, $x, y \ge 0$

[Dhaka Residential Model College, Dhaka]

a. If a, b $\in$ R then prove that $|a + b| \le |a| + |b|$. 2

b. If $\sqrt[3]{a + ib} = x + iy$ then show that $\frac{a}{x} - \frac{b}{y} = -2(x^2 + y^2)$. 4

c. Find the minimum value of $z = 2y - x$ with the help of graph of scenario-2. 4

Solution to the question no. 18

a See the Question No: 3(b), Chapter-1.

b Given that, $\sqrt[3]{a + ib} = x + iy$

or, $a + ib = (x + iy)^3$

or, $a + ib = x^3 + 3.x^2.iy + 3.x.(iy)^2 + (iy)^3$

$= x^3 + i3x^2y - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$ [$\because i^3 = -i$]

Equating real and imaginary parts, we get,

$\therefore$ $a = x^3 - 3xy^2$ and $b = 3x^2y - y^3$

or, $a = x(x^2 - 3y^2)$ and $b = y(3x^2 - y^2)$

or, $\frac{a}{x} = x^2 - 3y^2$ or, $\frac{b}{y} = 3x^2 - y^2$

Now, $\frac{a}{x} - \frac{b}{y} = x^2 - 3y^2 - 3x^2 + y^2$

$= -2x^2 - 2y^2 = -2(x^2 + y^2)$

$\frac{a}{x} - \frac{b}{y} = -2(x^2 + y^2)$ **(Shown)**

c The objective function, $Z_{min} = -x + 2y$

Subject to : $-x + 3y \le 10$

$x + y \le 6$

$x - y \le 2$

$x \ge 0, y \ge 0$

The corresponding equations of the constraint inequalities are :

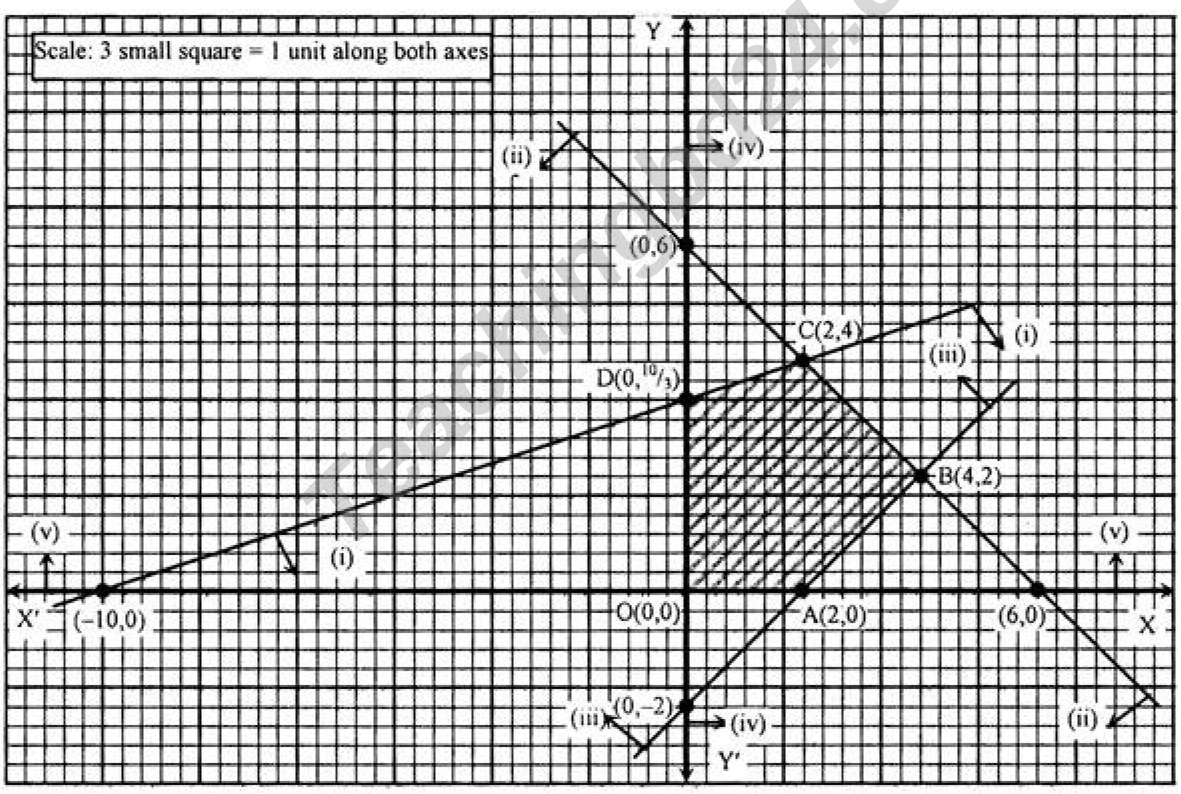
$-x + 3y = 10 \Rightarrow \frac{x}{-10} + \frac{y}{\frac{10}{3}} = 1$ ……… (i)

$x + y = 6 \Rightarrow \frac{x}{6} + \frac{y}{6} = 1$ ……… (ii)

$x - y = 2 \Rightarrow \frac{x}{2} + \frac{y}{-2} = 1$ ……… (iii)

$x = 0$ ……… (iv)

$y = 0$ ……… (v)



Now, we draw the graph of the equations (i) to (v) and determine the feasible region formed by the constraint inequalities.

In the figure, the shaded area indicates the feasible region whose vertices are O(0, 0), A(2, 0), B(4, 2), C(2, 4) and D$\left(0, \frac{10}{3}\right)$.

Now at, O(0, 0), z = 0

At A(2, 0), z = –2 + 0 = – 2

At B(4, 2), z = – 4 + (2 × 2) = 0

At C(2, 4), z = – 2 + (2 × 4) = 6

And at D$\left(0, \frac{10}{3}\right)$, $z = -0 + \left(2 \times \frac{10}{3}\right) = 6.67$

Clearly, the minimum value of Z Occurs at A(2, 0).

$\therefore$ $Z_{min} = -2$ **(Ans.)**

Ques ▶ 19 (i) $f(x) = x - 1$ (ii) $g(x) = a + bx + cx^2$

[Dhaka Residential Model College, Dhaka]

a. From (i) show that $\sqrt{f(3)}$ is an irrational number. 2

b. From (i) if $|f(x)| < \frac{1}{10}$ the show that $|f(x)f(x + 2)| < \frac{21}{100}$ 4

c. From (ii) if w is an imaginary cubic root of unity and $a + b + c = 0$ then prove that $\{g(w)\}^3 + \{g(w^2)\}^3 = 27abc$ 4

Solution to the question no. 19

a If possible, consider $\sqrt{2}$ is a rational number and $\sqrt{2} = \frac{p}{q}$, where p and q are positive integer and prime to each other.

By squaring both sides we get, $2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$ ... ... (i)

Since q is an integer, so q^2 is also an integer and $2q^2$ is an even number.
Then p^2 is an even number *i.e.* p is an even number.
Let $p = 2m$, m is any integer.
From (i), we get
$(2m)^2 = 2q^2 \Rightarrow 4m^2 = 2q^2 \Rightarrow q^2 = 2m^2$
Which shows that, q^2 is an even number, *i.e.* q is an even number.
But it is impossible, because p and q are prime to each other. i.e. there is no factor in common except 1.
Which is a contradiction.
Thus, $\sqrt{2}$ is not a rational number. Hence $\sqrt{2}$ is an irrational number.

b See the Question No: 1(b), Chapter-1

c Given that, $g(x) = a + bx + cx^2$

$\therefore g(\omega) = a + b\omega + c\omega^2$

and $g(\omega^2) = a + b\omega^2 + c(\omega^2)^2$
$= a + b\omega^2 + c\omega^4$
$= a + b\omega^2 + c\omega$

Let, $a + b\omega + c\omega^2 = p$ and $a + b\omega^2 + c\omega = q$

$\therefore$ L.H.S. $= p^3 + q^3$
$= (p + q)(p^2 - pq + q^2)$
$= (p + q)\{p^2 + (-1)pq + q^2\}$
$= (p + q)\{p^2 + (\omega^2 + \omega)pq + q^2\}$ $[\because \omega^2 + \omega = -1]$
$= (p + q)(p^2 + \omega^2 pq + \omega pq + q^2)$
$= (p + q)(p^2 + \omega pq + \omega^2 pq + q^2\omega^3)$ $[\because \omega^3 = 1]$
$= (p + q)\{p(p + \omega q) + \omega^2 q(p + \omega q)\}$
$= (p + q)(p + \omega q)(p + \omega^2 q)$

Now, $p + q = a + b\omega + c\omega^2 + a + b\omega^2 + c\omega$
$= 2a + b(\omega + \omega^2) + c(\omega^2 + \omega) = 2a - b - c$

$p + \omega q = a + b\omega + c\omega^2 + a\omega + b\omega^3 + c\omega^2$
$= a(1 + \omega) + b(\omega + 1) + 2c\omega^2$
$= a(-\omega^2) + b(-\omega^2) + 2c\omega^2$
$= \omega^2(2c - a - b)$

$p + \omega^2 q = a + b\omega + c\omega^2 + a\omega^2 + b\omega^4 + c\omega^3$
$= a + b\omega + c\omega^2 + a\omega^2 + b\omega + c$
$= a(1 + \omega^2) + 2b\omega + c(1 + \omega^2)$
$= a(-\omega) + 2b\omega + c(-\omega)$
$= \omega(2b - a - c)$

$\therefore$ $(p + q)(p + \omega q)(p + \omega^2 q)$
$= (2a - b - c)\omega^3(2c - a - b)(2b - a - c)$
$= (2a - b - c)(2c - a - b)(2b - a - c)$
$= \{2a - (b + c)\}\{2c - (a + b)\}\{2b - (a + c)\}$
$= \{2a - (-a)\}\{2c - (-c)\}\{2b - (-b)\}$ $[\because a + b + c = 0]$
$= 3a.3c.3b = 27abc =$ R.H.S.

$\therefore$ $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = 27abc$

$\therefore$ $\{g(\omega)\}^3 + \{g(\omega^2)\}^3 = 27$ abc **(Proved)**

Ques ▸ 20 $z_1 = a + ib$ and $z_2 = x + iy$ are two complex numbers. $f(\omega) = p + q\omega + r\omega^2$ where ω is an imaginary cubic root of unity and $p + q + r = 0$.

[Adamjee Cantonment College, Dhaka]

a. Find the value of $\sqrt[3]{-1}$. 2

b. If $\sqrt[3]{z_1} = z_2$ then show that, $\sqrt[3]{\bar{z}_1} = \bar{z}_2$. 4

c. Show that, $\{f(\omega)\}^3 + \{f(\omega)^2\}^3 = 27$ pqr. 4

Solution to the question no. 20

a Let, $x = \sqrt[3]{-1}$

or, $x^3 = -1$

or, $x^3 + 1 = 0$

or, $(x + 1)(x^2 - x + 1) = 0$

Either, $x + 1 = 0$

$\therefore$ $x = -1$

or, $x^2 - x + 1 = 0$

or, $x = \frac{-(-1) \pm \sqrt{1 - 4}}{2}$

$= \frac{1 \pm \sqrt{-3}}{2}$

$\therefore$ The values of $\sqrt[3]{-1}$ are, $-1, \frac{1 + \sqrt{-3}}{2}, \frac{1 - \sqrt{-3}}{2}$ **(Ans.)**

b Given that, $z_1 = a + ib$ and $z_2 = x + iy$

Here, $\sqrt[3]{z_1} = z_2$

or, $\sqrt[3]{a + ib} = x + iy$

or, $a + ib = (x + iy)^3$ [Taking cube on both sides]

or, $a + ib = x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3$

or, $a + ib = x^3 + i3x^2y - 3xy^2 - iy^3$ $[\because i^2 = -1]$

$\therefore$ $a + ib = x^3 - 3xy^2 + i(3x^2y - y^3)$

Equating real and imaginary parts, we get,
$a = x^3 - 3xy^2$ and $b = 3x^2y - y^3$

Now, $a - ib = x^3 - 3xy^2 - i(3x^2y - y^3)$
$= x^3 - 3xy^2 - i3x^2y + iy^3$
$= x^3 - i3x^2y - 3xy^2 - i^3y^3$ $[\because i^2 = -1]$
$= (x)^3 - 3.x^2.iy + 3.x(iy)^2 - (iy)^3$

or, $a - ib = (x - iy)^3$

or, $\sqrt[3]{a - ib} = x - iy$

or, $\sqrt[3]{\overline{a + ib}} = \overline{x + iy}$

$\therefore$ $\sqrt[3]{\bar{z}_1} = \bar{z}_2$ **(Shown)**

c Similar to Question No: 19(c).

Ques ▸ 21 Stem-1 : $\sqrt[3]{a + ib} = x + iy$ and Stem-2 : $f(x) = x + 2$

[BAF Shaheen College, Dhaka]

a. Find the solution set: $\left|\frac{1}{x - 3}\right| \geq 2; x \neq 3.$ 2

b. From the stem-1, prove that $\sqrt[3]{a + ib} = x + iy$. 4

c. From the stem-2, find the maximum value of $Z = f(2x) + f(3y - 4)$ under the conditions: $x + 2y \leq 10, x + y \leq 6, x \leq 4, x, y \geq 0.$ 4

Solution to the question no. 21

a $\left|\frac{1}{x - 3}\right| \geq 2$ $[x \neq 3]$

or, $|x - 3| \leq \frac{1}{2}$ or, $-\frac{1}{2} \leq x - 3 \leq \frac{1}{2}$

or, $-\frac{1}{2} + 3 \leq x \leq \frac{1}{2} + 3$ or, $\frac{5}{2} \leq x \leq \frac{7}{2}$

$\therefore$ The required solution $\frac{5}{2} \le x \le \frac{7}{2}$ and $x \ne 3$

$\therefore$ Th solution set, $S = \left\{ x \in \mathbb{R} : \frac{5}{2} \le x \le \frac{7}{2} \text{ and } x \ne 3 \right\}$ **(Ans.)**

b Given that, $\sqrt[3]{a + ib} = x + iy$

or, $a + ib = (x + iy)^3$ [Taking cube on both sides]

or, $a + ib = x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3$

or, $a + ib = x^3 + i3x^2y - 3xy^2 - iy^3$ $\quad [\because i^2 = -1]$

$\therefore$ $a + ib = x^3 - 3xy^2 + i(3x^2y - y^3)$

Equating real and imaginary parts from both sides,

$a = x^3 - 3xy^2$ and $b = 3x^2y - y^3$

Now, $a - ib = x^3 - 3xy^2 - i(3x^2y - y^3)$

$= x^3 - 3xy^2 - i3x^2y + iy^3$

$= x^3 - i3x^2y - 3xy^2 - i^3y^3$ $[\because i^2 = -1]$

$= (x)^3 - 3.x^2.iy + 3.x.(iy)^2 - (iy)^3$

or, $a - ib = (x - iy)^3$

$\therefore$ $\sqrt[3]{a - ib} = x - iy$ **(Shown).**

c Given that, $f(x) = x + 2$

Objective function,

$Z_{max} = f(2x) + f(3y - 4) = 2x + 2 + 3y - 4 + 2$

$= 2x + 3y$

Subject to, $x + 2y \le 10$, $x + y \le 6$, $x \le 4$, $x, y \ge 0$

Taking corresponding equations of the constraint inequalities, we sketch the graph of them and determine the feasible region of the solution set. Therefore, we get,

$x + 2y = 10$

$\Rightarrow \frac{x}{10} + \frac{y}{5} = 1$ ……… (i)

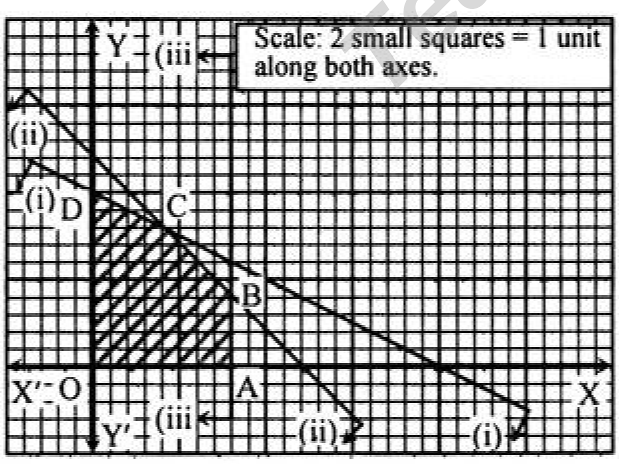
$x + y = 6$

$\Rightarrow \frac{x}{6} + \frac{y}{6} = 1$ ……… (ii)

$x = 4$ ……… (iii)

$x = 0$ ……… (iv)

$y = 0$ ……… (v)



In the graph, the shaded area indicates the feasible region determined by the constraint inequalities. The corner points of the feasible region are,

O(0, 0), A(4, 0), B(4, 2), C(2, 4) and D(0, 5)

Now, at O(0, 0), $z = 2 \times 0 + 3 \times 0 = 0$

At A(4, 0), $z = 2 \times 4 + 3 \times 0 = 8$

At B(4, 2), $z = 2 \times 4 + 3 \times 2 = 14$

At C(2, 4), $z = 2 \times 2 + 3 \times 4 = 16$

At D(0, 5), $z = 2 \times 0 + 3 \times 5 = 15$

Clearly, z attains it maximum value at C(2, 4)

$\therefore$ The, maximum value, $Z_{max} = 16$ **(Ans.)**

Ques ▶ 22 $f(x, y) = x + iy$

[Chattogram Cantonment Public College, Chattogram]

a. Find the modulus and argument of $-1 + \sqrt{3}i$. 2

b. In the stem if $x = -8$, $y = -6$ then find the value of $\sqrt{f}$. 4

c. If $\sqrt[3]{f(a,b)} = a + ib$ then prove that $\sqrt[3]{f(a, -b)} = a - ib$. 4

Solution to the question no. 22

a Let, $z = 1 + \sqrt{3}i$

$\therefore$ The modulus of $z = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

and the argument of $z = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

But z lies in the 2nd quadrant and so the argument $= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. **(Ans.)**

b Given that,

$f(x, y) = x + iy$

$\therefore f(-8, -6) = -8 - 6i$

$\therefore \sqrt{f} = \sqrt{-8 - 6i}$

$= \sqrt{1 - 9 - 6i}$

$= \sqrt{1^2 - 2.\,1.\,3i + (3i)^2}$

$= \sqrt{(1 - 3i)^2}$

$= (1 - 3i)$ **(Ans.)**

c Given that,

$\sqrt[3]{f(a, b)} = a + ib$

or, $\sqrt[3]{a + ib} = a + ib$

or, $a + ib = (a + ib)^3$

or, $a + ib = a^3 + 3ia^2b - 3ab^2 - ib^3$ $[\because i^2 = -1]$

$\therefore a + ib = a^3 - 3ab^2 + i(3a^2b - b^3)$

Equating real and imaginary parts from both sides,

$a = a^3 - 3ab^2$ and $b = 3a^2b - b^3$

Now, $a - ib = (a^3 - 3ab^2) - i(3a^2b - b^3)$

$= a^3 - 3ia^2b - 3ab^2 + ib^3$

$= a^3 - 3ia^2b + 3a(ib)^2 - (ib)^3$

$= (a - ib)^3$

$\therefore \sqrt[3]{a - ib} = a - ib$. **(Proved)**