Chapter-2: Algebraic Expressions

Question ▶ 1 $F(x) = \sqrt{1-2x}$ and $Q(x) = \dfrac{x^2}{x^2-16}$ [All Board-18]

a. Find the domain of F (x). 2
b. Determine whether $F^{-1}(x)$ is one-one or not. 4
c. Express Q(x) into partial fractions. 4

Solution to the question no. 1

a Given, $F(x) = \sqrt{1-2x}$

The function F(x) will be defined if and only if,

$1 - 2x \geq 0$

Or, $-2x \geq -1$ or, $2x \leq 1$

$\therefore x \leq \dfrac{1}{2}$

$\therefore$ Domain of $F(x) = \sqrt{1-2x}$ is $\{x \in \mathbb{R} : x \leq \frac{1}{2}\}$ **(Ans.)**

b Let, $y = F(x) = \sqrt{1-2x}$

Or, $y^2 = 1 - 2x$ [by squaring]

Or, $2x = 1 - y^2$

Or, $x = \dfrac{1-y^2}{2} = F^{-1}(y)$

$\therefore F^{-1}(y) = \dfrac{1-y^2}{2}$

$\therefore F^{-1}(x) = \dfrac{1-x^2}{2}$

Let, $x_1, x_2 \in$ Dom $F^{-1}(x)$

The function $F^{-1}(x)$ will be one-one if and only if for $F(x_1) = F(x_2)$ implies $x_1 = x_2$

Let, $F^{-1}(x_1) = F^{-1}(x_2)$

Or, $\dfrac{1-x_1^2}{2} = \dfrac{1-x_2^2}{2}$

Or, $1 - x_1^2 = 1 - x_2^2$

$\therefore x_1 = x_2$ [$\because$ Range of F(x) will be domain of $F^{-1}(x)$. Again, $F(x) \geq 0$, so, $x_1, x_2 \geq 0$.]

$\therefore$ The function is one-one. **(Ans.)**

c Given,

$Q(x) = \dfrac{x^2}{x^2-16} = \dfrac{x^2-16+16}{x^2-16} = \dfrac{x^2-16}{x^2-16} + \dfrac{16}{x^2-16}$

$= 1 + \dfrac{16}{(x+4)(x-4)}$

Let, $\dfrac{16}{(x+4)(x-4)} \equiv \dfrac{A}{(x+4)} + \dfrac{B}{(x-4)}$ ...... (i)

Multiplying (i) by $(x+4)(x-4)$ we get,

$16 \equiv A(x-4) + B(x+4)$ ...... (ii)

This is true for any values of x.

Putting $x = -4$ in (ii) we get,

$16 = A(-4-4) + B(-4+4)$

Or, $16 = -8A$

$\therefore A = -2$

Again, putting $x = 4$ in (ii) we get,

$16 = A(4-4) + B(4+4)$

Or, $16 = 8B$

$\therefore B = 2$

Putting the values of A and B in (i) we get,

$\dfrac{16}{(x+4)(x-4)} = \dfrac{-2}{x+4} + \dfrac{2}{x-4}$

$\therefore Q(x) = \dfrac{x^2}{x^2-16} = 1 - \dfrac{2}{(x+4)} + \dfrac{2}{(x-4)}$; These are the required partial fractions.

Question ▶ 2 $P(x) = 18x^3 + 15x^2 - x + a$, $Q(x) = x^3 + x^2 - 6x$ are two algebraic equation. [C.B.17]

a. Resolve into factors of Q (x). 2
b. Find the value of a if (3x + 2) is a factor of the polynomial P (x). 4
c. Express the partial fractions of $\dfrac{x^2+x-1}{Q(x)}$ 4

Solution to the question no. 2

a Given,

$Q(x) = x^3 + x^2 - 6x$
$= x(x^2 + x - 6)$
$= x(x^2 + 3x - 2x - 6)$
$= x\{x(x+3) - 2(x+3)\}$
$= x(x+3)(x-2)$ **(Ans.)**

b Given, $P(x) = 18x^3 + 15x^2 - x + a$

Since (3x + 2) is a factor of P(x). So if P(x) is divided by (3x + 2) the remainder will be zero. That is-

$P\left(-\dfrac{2}{3}\right) = 0$

Or, $18\left(-\dfrac{2}{3}\right)^3 + 15\left(-\dfrac{2}{3}\right)^2 - \left(-\dfrac{2}{3}\right) + a = 0$

Or, $-18 \cdot \dfrac{8}{27} + 15 \cdot \dfrac{4}{9} + \dfrac{2}{3} + a = 0$

Or, $-\dfrac{16}{3} + \dfrac{20}{3} + \dfrac{2}{3} + a = 0$

Or, $a = \dfrac{16}{3} - \dfrac{20}{3} - \dfrac{2}{3}$

Or, $a = \dfrac{16-20-2}{3}$

Or, $a = \dfrac{-6}{3}$

$\therefore a = -2$ **(Ans.)**

c $\dfrac{x^2+x-1}{Q(x)} = \dfrac{x^2+x-1}{x^3+x^2-6x}$

$= \dfrac{x^2+x-1}{x(x^2+x-6)}$

$= \dfrac{x^2+x-1}{x(x-2)(x+3)}$

Let, $\dfrac{x^2+x-1}{x(x-2)(x+3)} \equiv \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+3}$ ....(i)

Multiplying both sides by $x(x-2)(x+3)$ we get,

$x^2 + x - 1 \equiv A(x-2)(x+3) + Bx(x+3) + Cx(x-2)$ ...... (ii)

Putting the value x = 0 in (ii) we get,

$-1 = A.(-2).3$

Or, $-6A = -1$

$\therefore A = \dfrac{1}{6}$

Putting the value x = 2 in (ii) we get,

$2^2 + 2 - 1 = A(2-2)(2+3) + B.2.(2+3) + C.2(2-2)$

Or, $4 + 2 - 1 = B.2.5$

Or, $10B = 5$

$\therefore B = \frac{1}{2}$

Putting the value x = – 3 in (ii) we get,

$(-3)^2 + (-3) - 1 = A(-3-2)(-3+3) + B.(-3)(-3+3) + C.(-3)(-3-2)$

Or, $9 - 3 - 1 = C.(-3)(-5)$

Or, $15C = 5$

$\therefore C = \frac{1}{3}$

Now, putting the values of A, B and C in (i) we get,

$$\frac{x^2 - x - 1}{x(x-2)(x+3)} = \frac{\frac{1}{6}}{x} + \frac{\frac{1}{2}}{x-2} + \frac{\frac{1}{3}}{x+3}$$
$$= \frac{1}{6x} + \frac{1}{2(x-2)} + \frac{1}{3(x+3)}$$

Which is required partial fraction.

Question ▶ 3 $P(x) = x^3 - 6x^2 + 11x - 6$. [J.B.17]

a. Determine the ratio of degree and leading co-efficient of P(x). 2

b. If the remainders of P(x) upon division by x – m and x – n are same where m ≠ n, then show that, $m^2 + mn + n^2 - 6m - 6n + 11 = 0$. 4

c. Express $\frac{x^3}{P(x)}$ as partial fractions. 4

Solution to the question no. 3

a Given, $P(x) = x^3 - 6x^2 + 11x - 6$

Here, maximum power of variable x = 3

∴ The degree of P(x) = 3

and leading coefficient of P(x) = 1

∴ The ratio of degree and leading coefficient of P(x) = 3 : 1 **(Ans.)**

b If P(x) is divided by (x – m) the remainder will be P(m)

∴ $P(m) = m^3 - 6m^2 + 11m - 6$

and if P(x) is divided by (x – n) the remainder will be P(n)

∴ $P(n) = n^3 - 6n^2 + 11n - 6$

According to question, P(m) = P(n)

Or, $m^3 - 6m^2 + 11m - 6 = n^3 - 6n^2 + 11n - 6$

Or, $m^3 - n^3 - 6m^2 + 6n^2 + 11m - 11n = 0$

Or, $(m-n)(m^2 + mn + n^2) - 6(m+n)(m-n) + 11(m-n) = 0$

Or, $(m-n)(m^2 + mn + n^2 - 6m - 6n + 11) = 0$

∴ $m^2 + mn + n^2 - 6m - 6n + 11 = 0$ [∵ m ≠ n, so m – n ≠ 0] **(Shown)**

c $P(x) = x^3 - 6x^2 + 11x - 6$

$= x^3 - x^2 - 5x^2 + 5x + 6x - 6$

$= x^2(x-1) - 5x(x-1) + 6(x-1)$

$= (x-1)(x^2 - 5x + 6)$

$= (x-1)(x-2)(x-3)$

$\therefore \frac{x^3}{P(x)} = \frac{x^3}{(x-1)(x-2)(x-3)}$

After that, see example-4 of exercise-2 from your textbook. Page-58

Question ▶ 4 $f(x) = 18x^3 + 15x^2 - x + c$, $g(x) = x^2 - 4x - 7$ and $h(x) = x^3 - x^2 - 10x - 8$ are three polynomials of variable x. [B.B.17]

a. Resolve h(x) into factors. 2

b. If (3x + 2) is a factor of f(x), find the value of c. 4

c. Express $\frac{g(x)}{h(x)}$ as partial fractions. 4

Solution to the question no. 4

a Given, $h(x) = x^3 - x^2 - 10x - 8$

If x = –1 then h(–1) = 0.

Therefore (x + 1) is a factor of h(x).

∴ $h(x) = x^3 - x^2 - 10x - 8$

$= x^3 + x^2 - 2x^2 - 2x - 8x - 8$

$= x^2(x+1) - 2x(x+1) - 8(x+1)$

$= (x+1)(x^2 - 2x - 8)$

$= (x+1)(x^2 - 4x + 2x - 8)$

$= (x+1)\{x(x-4) + 2(x-4)\}$

$= (x+1)(x+2)(x-4)$ **(Ans.)**

b Given, $f(x) = 18x^3 + 15x^2 - x + c$

Since (3x + 2) is a factor of f(x), according to inverse theorem of factor theorem, $f\left(-\frac{2}{3}\right) = 0$

Here, $f\left(-\frac{2}{3}\right) = 18\left(-\frac{2}{3}\right)^3 + 15\left(-\frac{2}{3}\right)^2 - \left(-\frac{2}{3}\right) + c$

$= -18.\frac{8}{27} + 15.\frac{4}{9} + \frac{2}{3} + c$

$= -\frac{16}{3} + \frac{20}{3} + \frac{2}{3} + c$

$= \frac{-16 + 20 + 2 + 3c}{3} = \frac{6 + 3c}{3}$

According to condition, $f\left(-\frac{2}{3}\right) = 0$

Or, $\frac{6 + 3c}{3} = 0$

Or, $6 + 3c = 0$

∴ $c = -2$ **(Ans.)**

c Given, $g(x) = x^2 - 4x - 7$

From 'a' we get,

$h(x) = (x+1)(x+2)(x-4)$

$\therefore \frac{g(x)}{h(x)} = \frac{x^2 - 4x - 7}{(x+1)(x+2)(x-4)}$

$\therefore \frac{x^2 - 4x - 7}{(x+1)(x+2)(x-4)}$ is a proper fraction.

Let, $\frac{x^2 - 4x - 7}{(x+1)(x+2)(x-4)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-4}$ ...... (i)

Multiplying both sides of (i) by (x + 1) (x + 2) (x – 4) we get,

$x^2 - 4x - 7 \equiv A(x+2)(x-4) + B(x+1)(x-4) + C(x+1)(x+2)$ .... (ii)

The equation (ii) is true for all values of x.

Putting x = – 1 in equation (ii) we get,

$1 + 4 - 7 = A(-1+2)(-1-4)$

Or, $-2 = A(-5)$ ∴ $A = \frac{2}{5}$

Putting x = – 2 in equation (ii) we get,

$4 + 8 - 7 = B(-2+1)(-2-4)$

Or, $5 = B(-1)(-6)$ ∴ $B = \frac{5}{6}$

Putting x = 4 in equation (ii) we get,

$16 - 16 - 7 = C(4+1)(4+2)$

Or, $-7 = C(5)(6)$ ∴ $C = -\frac{7}{30}$

Now putting the values of A, B, C in equation (i) we get,

$$\frac{x^2-4x-7}{(x+1)(x+2)(x-4)} \equiv \frac{2}{5(x+1)} + \frac{5}{6(x+2)} - \frac{7}{30(x-4)}$$

This is the expression of the given fraction into the partial fraction.

Question ▶ 5 $f(a) = a^3 + 5a^2 + 6a + 8$ and

$g(a) = \frac{2a}{(a+1)(a^2+1)^2}$ are two algebraical expressions. [D.B. 16]

a. Find the value of $f(-3)$. 2

b. If $f(a)$ yields the same remainder upon division by $x - p$ and $x - q$ where $p \neq q$, show that $p^2 + q^2 + pq + 5p + 5q + 6 = 0$. 4

c. Express g(a) as a sum of partial fractions. 4

Solution to the question no. 5

a Given,

$f(a) = a^3 + 5a^2 + 6a + 8$

$\therefore f(-3) = (-3)^3 + 5.(-3)^2 + 6.(-3) + 8$

$= -27 + 45 - 18 + 8$

$= 53 - 45$

$= 8$ **(Ans.)**

b Given,

$f(a) = a^3 + 5a^2 + 6a + 8$

If we divide f(a) by $(x - p)$ the remainder will be f(p)

$\therefore f(p) = p^3 + 5p^2 + 6p + 8$

Again, If we divide f(a) by $(x - q)$ the remainder will be f(q).

$\therefore f(q) = q^3 + 5q^2 + 6q + 8$

According to the question f(p) = f(q)

Or, $p^3 + 5p^2 + 6p + 8 = q^3 + 5q^2 + 6q + 8$

Or, $p^3 - q^3 + 5(p^2 - q^2) + 6(p - q) = 0$

Or, $(p - q)(p^2 + pq + q^2) + 5(p + q)(p - q) + 6(p - q) = 0$

Or, $(p - q)(p^2 + pq + q^2 + 5p + 5q + 6) = 0$

Since $p \neq q$, therefore $p - q \neq 0$

$\therefore p^2 + pq + q^2 + 5p + 5q + 6 = 0$ **(Showed)**

c Given,

$$g(a) = \frac{2a}{(a+1)(a^2+1)^2}$$

$$\frac{2a}{(a+1)(a^2+1)^2} \equiv \frac{A}{a+1} + \frac{Ba+C}{a^2+1} + \frac{Da+E}{(a^2+1)^2} \text{ ..... (i)}$$

Multiplying both sides by $(a+1)(a^2+1)^2$ in equation (i) we get,

$2a \equiv A(a^2+1)^2 + (Ba+C)(a+1)(a^2+1) + (Da+E)(a+1)$ ...... (ii)

Substituting the value of $a = -1$ in equation (ii) we get,

$2(-1) = A(1+1)^2$

Or, $-2 = 4A$

Or, $A = -\frac{2}{4}$

$\therefore A = -\frac{1}{2}$

From (ii)

$2a \equiv A(a^4 + 2a^2 + 1) + (Ba + C)(a^3 + a + a^2 + 1) + Da^2 + Da + Ea + E$

Or, $2a \equiv Aa^4 + 2Aa^2 + A + Ba^4 + Ba^2 + Ba^3 + Ba + Ca^3 + Ca + Ca^2 + C + Da^2 + Da + Ea + E$

$\therefore \quad 2a = (A + B)a^4 + (B + C)a^3 + (2A + B + C + D)a^2 + (B + C + D + E)a + (A + C + E)$

Equating the coefficients of a^4, a^3, a^2 and a we get,

$A + B = 0$

Or, $-\frac{1}{2} + B = 0$ [$\because A = -\frac{1}{2}$]

$\therefore B = \frac{1}{2}$

Again, $B + C = 0$

Or, $\frac{1}{2} + C = 0$

$\therefore C = -\frac{1}{2}$

Again, $2A + B + C + D = 0$

Or, $2.\left(-\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{2} + D = 0$

Or, $-1 + D = 0$

$\therefore D = 1$

Again, $B + C + D + E = 2$

Or, $\frac{1}{2} - \frac{1}{2} + 1 + E = 2$

Or, $E = 2 - 1$

$\therefore E = 1$

Now, substituting the values of A, B, C, D and E in (i) we get,

$$\frac{2a}{(a+1)(a^2+1)^2} = \frac{-\frac{1}{2}}{a+1} + \frac{\frac{1}{2}a - \frac{1}{2}}{a^2+1} + \frac{a+1}{(a^2+1)^2}$$

$$= -\frac{1}{2(a+1)} + \frac{a-1}{2(a^2+1)} + \frac{a+1}{(a^2+1)^2}$$

Which is the desired partial fractions.

Question ▶ 6 $f(x) = \frac{1}{\sqrt{3x-1}}$ and $g(x) = \frac{x^2}{x^2-16}$ are two functions. [R.B.16]

a. Find the domain of the function stated by $f(x)$. 2

b. Determine $f^{-1}(-1)$. 4

c. Express g(x) as partial fractions. 4

Solution to the question no. 6

a Given, $f(x) = \frac{1}{\sqrt{3x-1}}$

Now, the function $f(x)$ will be difined if and only if $3x - 1 > 0$.

Or, $3x - 1 + 1 > 0 + 1$

Or, $3x > 1$

$\therefore x > \frac{1}{3}$

$\therefore$ The domain of the function = $\{x \in \mathbb{R} : x > \frac{1}{3}\}$ **(Ans.)**

b Let,

$y = f(x) = \frac{1}{\sqrt{3x-1}}$

$\therefore y = \frac{1}{\sqrt{3x-1}}$

Or, $y^2 = \frac{1}{3x-1}$ [Squaring both sides]

Or, $3x - 1 = \frac{1}{y^2}$

Or, $3x = \frac{1}{y^2} + 1$

Or, $3x = \frac{1 + y^2}{y^2}$

Or, $x = \frac{1 + y^2}{3y^2}$

Or, $f^{-1}(y) = \frac{1+y^2}{3y^2}$ $[f(x) = y \therefore x = f^{-1}(y)]$

$\therefore f^{-1}(-1) = \frac{1+(-1)^2}{3(-1)^2} = \frac{1+1}{3.1} = \frac{2}{3}$ **(Ans.)**

c Given,

$g(x) = \frac{x^2}{x^2-16} = \frac{x^2-16+16}{x^2-16} = \frac{x^2-16}{x^2-16} + \frac{16}{x^2-16}$

$= 1 + \frac{16}{(x+4)(x-4)}$

Suppose, $\frac{16}{(x+4)(x-4)} \equiv \frac{A}{(x+4)} + \frac{B}{(x-4)}$ .... (i)

Multiplying both sides by $(x+4)(x-4)$ in (i)

$16 \equiv A(x-4) + B(x+4)$ .... (ii)

Which is true for all values of x.

Substituting the value of $x = -4$ in (i) we get,

$16 \equiv A(-4-4) + B(-4+4)$

Or, $16 = -8A$

$A = -2$

Again, substituting the value of $x = 4$ in (ii) we get,

$16 = A(4-4) + B(4+4)$

Or, $16 = 8B$

$\therefore B = 2$

Substituting the values of A and B in (i)

$\frac{16}{(x+4)(x-4)} = \frac{-2}{x+4} + \frac{2}{x-4}$

$\therefore g(x) = \frac{x^2}{x^2-16} = 1 - \frac{2}{(x+4)} + \frac{2}{(x-4)}$ This is the expression of the given fraction into pertial fraction. **(Ans.)**

Question ▶7 $P(x) = x^2 + x - 12$, $Q(x) = 9x + 2$. [Dj. B.16]

a. Find the domain $F(x) = \frac{2x}{x+3}$. 2

b. If P(x) yields the same remainder upon division by $2x - a$ and $2x - b$ where $a \neq b$, show that, $a + b + 2 = 0$ 4

c. Express $\frac{Q(x)}{P(x)}$ as partial fractions. 4

Solution to the question no. 7

a Given, $F(x) = \frac{2x}{x+3}$

If $x + 3 = 0$

Or, $x = -3$, the function f(x) does not exist.

$\therefore$ Domain, $F = \mathbb{R} - \{-3\}$ **(Ans.)**

b Given, $P(x) = x^2 + x - 12$

If we divide P(x) by $(2x - a)$ the remainder will be $P\left(\frac{a}{2}\right)$

$\therefore P\left(\frac{a}{2}\right) = \left(\frac{a}{2}\right)^2 + \frac{a}{2} - 12$

Again, if we divide P(x) by $(2x - b)$, the remainder will be $P\left(\frac{b}{2}\right)$.

$\therefore P\left(\frac{b}{2}\right) = \left(\frac{b}{2}\right)^2 + \frac{b}{2} - 12$

According to the questions,

$\left(\frac{a}{2}\right)^2 + \frac{a}{2} - 12 = \left(\frac{b}{2}\right)^2 + \frac{b}{2} - 12$

Or, $\frac{a^2}{4} + \frac{a}{2} - 12 = \frac{b^2}{4} + \frac{b}{2} - 12$

Or, $a^2 + 2a - 48 = b^2 + 2b - 48$

Or, $a^2 + 2a - 48 - b^2 - 2b + 48 = 0$

Or, $a^2 - b^2 + 2a - 2b = 0$

Or, $(a+b)(a-b) + 2(a-b) = 0$

Or, $(a+b+2)(a-b) = 0$

Since $a \neq b$

$\therefore a + b + 2 = 0$ **(Shown)**

c Given,

$P(x) = x^2 + x - 12$

and $Q(x) = 9x + 2$

Now, $x^2 + x - 12 = x^2 + 4x - 3x - 12$

$= x(x+4) - 3(x+4)$

$= (x+4)(x-3)$

$\therefore \frac{Q(x)}{P(x)} = \frac{9x+2}{x^2+x-12} = \frac{9x+2}{(x+4)(x-3)}$

Suppose, $\frac{9x+2}{(x+4)(x-3)} \equiv \frac{A}{x+4} + \frac{B}{x-3}$ .... (i)

Multiplying both sides by of equation (i) with $(x+4)(x-3)$ we get,

$9x + 2 \equiv A(x-3) + B(x+4)$ .... (ii)

Which is true for all values of x

Now, Putting $x = -4$ both sides in equation (ii),

$9(-4) + 2 = A(-4-3) + B(-4+4)$

Or, $-36 + 2 = -7A$

Or, $-34 = -7A$

$\therefore A = \frac{34}{7}$

Again, putting $x = 3$ in both sides of equation (ii) we get,

$9.3 + 2 = A(3-3) + B(3+4)$

Or, $29 = 7B$

$\therefore B = \frac{29}{7}$

Putting the values of A and B in equation (i)

$\frac{9x+2}{(x+4)(x-3)} = \frac{\frac{34}{7}}{x+4} + \frac{\frac{29}{7}}{x-3}$

$\therefore \frac{Q(x)}{P(x)} = \frac{34}{7(x+4)} + \frac{29}{7(x-3)}$ which is the desired partial fractions.

Question ▶8 $x - 1$ is a factor of the polynomial $g(x) = px^3 + qx^2 + rx + s$ and all coefficients of the polynomial are integers and $p \neq 0$, $s \neq 0$ another expression $Q(x) = \frac{x^3}{x^2-16}$. [S.B.16]

a. Show that, $p + q + r + s = 0$. 2

b. If $p = 1$, $q = 5$, $r = 6$ and $s = 8$ and g(x) yields the same remainder upon division by $x - k$ and $x - l$ where $k \neq 1$ then show that $k^2 + l^2 + kl + 5k + 5l + 6 = 0$. 4

c. Express Q(x) as partial fraction. 4

Solution to the question no. 8

a Given,

$g(x) = px^3 + qx^2 + rx + s$

$(x - 1)$ is a factor of g(x)

i.e. $g(1) = 0$

Or, $p.(1)^3 + q.(1)^2 + r.1 + s = 0$

$\therefore p + q + r + s = 0$ **(Shown)**

b Here,

$g(x) = px^3 + qx^2 + rx + s$

and $p = 1$, $q = 5$, $r = 6$, $s = 8$

$\therefore g(x) = x^3 + 5x^2 + 6x + 8$

If we divide g(x) by (x – k) and (x – l) the remainder will be g(k) and g(l) respectively

$g(k) = g(l)$

Or, $k^3 + 5k^2 + 6k + 8 = l^3 + 5l^2 + 6l + 8$

Or, $k^3 - l^3 + 5(k^2 - l^2) + 6(k - l) = 0$

Or, $(k - l)(k^2 + kl + l^2) + 5(k + l)(k - l) + 6(k - l) = 0$

Or, $(k - l)(k^2 + kl + l^2 + 5k + 5l + 6) = 0$

Since $k \neq l$, therefore $(k - l) \neq 0$

$\therefore k^2 + l^2 + kl + 5k + 5l + 6 = 0$. **(Shown)**

c Given,

$$Q(x) = \frac{x^3}{x^2 - 16}$$

$$= \frac{x(x^2 - 16) + 16x}{x^2 - 16x} = x + \frac{16x}{x^2 - 16}$$

$$= x + \frac{16x}{(x + 4)(x - 4)}$$

Suppose, $\frac{16x}{(x + 4)(x - 4)} \equiv \frac{A}{x + 4} + \frac{B}{x - 4}$ .... (i)

Multiplying both sides of (i) with (x + 4) (x – 4) in (i) we get,

$16x \equiv A(x - 4) + B(x + 4)$ .... (ii)

Which is true for all values of x.

Putting x = – 4 both sides in (ii) we get,

$16 \times (-4) = A(-4 - 4) + B(-4 + 4)$

Or, $-64 = -8A$

$\therefore A = 8$

Again, putting x = 4 both sides in (ii) we get,

$16 \times 4 = A(4 - 4) + B(4 + 4)$

Or, $64 = 8B$

$\therefore B = 8$

Putting the value of A and B in (i) we get,

$$\frac{16x}{(x + 4)(x - 4)} = \frac{8}{x + 4} + \frac{8}{x - 4} = 8\left(\frac{1}{x + 4} + \frac{1}{x - 4}\right)$$

$$\therefore Q(x) = x + 8\left(\frac{1}{x + 4} + \frac{1}{x - 4}\right);$$

This is the expression of the given fraction into partial fraction. **(Ans.)**

Question ▶ 9 $p(x) = x^3 + x^2 - 6x$ and $f(x) = x^2 - 9x - 6$ are two functions. [Ctg.B.16]

a. Find the remainder by using remainder theorem when $f(x)$ is divided by (x + 3). 2

b. If p(x) yields the same remainder upon division by (x – a) and (x – b) where $a \neq b$, then show that $a^2 + ab + b^2 + a + b = 6$. 4

c. Express $\frac{f(x)}{p(x)}$ into partial fractions. 4

Solution to the question no. 9

a Given,

$f(x) = x^2 - 9x - 6$

Now, if we divide $f(x)$ by (x + 3) the remainder will be $f(-3)$.

$\therefore f(-3) = (-3)^2 - 9(-3) - 6 = 9 + 27 - 6$

$= 30$ **(Ans.)**

b Given,

$p(x) = x^3 + x^2 - 6x$

If we divide p(x) by (x – a) and (x – b) the remainder will be p(a) and p(b) respectively.

According to the quention.

$p(a) = p(b)$

Or, $a^3 + a^2 - 6a = b^3 + b^2 - 6b$

Or, $a^3 - b^3 + a^2 - b^2 - 6(a - b) = 0$

Or, $(a - b)(a^2 + ab + b^2) + (a + b)(a - b) - 6(a - b) = 0$

Or, $(a - b)(a^2 + ab + b^2 + a + b - 6) = 0$

Or, $a^2 + ab + b^2 + a + b - 6 = 0$ [Since $a \neq b$ therefore $a - b \neq 0$]

$\therefore a^2 + ab + b^2 + a + b = 6$ **(Shown)**

c $\frac{f(x)}{p(x)} = \frac{x^2 - 9x - 6}{x^3 + x^2 - 6x} = \frac{x^2 - 9x - 6}{x(x^2 + x - 6)} = \frac{x^2 - 9x - 6}{x(x^2 + 3x - 2x - 6)}$

$\therefore \frac{f(x)}{p(x)} = \frac{x^2 - 9x - 6}{x(x - 2)(x + 3)}$

Suppose, $\frac{x^2 - 9x - 6}{x(x - 2)(x + 3)} \equiv \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 3}$ .... (i)

Multiplying both sides by x(x – 2) (x + 3) in (i) we get,

$x^2 - 9x - 6 \equiv A(x - 2)(x + 3) + Bx(x + 3) + Cx(x - 2)$ ..... (ii)

Which is true for all values of x.

Now putting the value of x = 0 is both sides of (ii) we get,

$0 - 0 - 6 = A(0 - 2)(0 + 3) + 0 + 0$

Or, $-6 = -6A$

$\therefore A = 1$

Again, putting the value of x = 2 both sides in (ii) we get,

$2^2 - 9.2 - 6 = 0 + B.2(2 + 3) + 0$

Or, $-20 = 10B$

$\therefore B = -2$

Putting the value of x = – 3 in both sides of (ii) we get,

$(-3)^2 - 9.(-3) - 6 = 0 + 0 + C(-3)(-3 - 2)$

Or, $30 = 15C$

$\therefore C = 2$

Putting the value of A, B, C in (i) we get,

$$\frac{x^2 - 9x - 6}{x(x - 2)(x + 3)} = \frac{1}{x} - \frac{2}{x - 2} + \frac{2}{x + 3}$$

This is the expression of the given fraction into partial fraction.

Question ▶ 10 $P(x) = x^3 - x^2 + ax + b$ and $Q(x) = x^2 - 2x - 8$. [J.B. 16]

a. Resolve Q(x) into factors. 2

b. Express $\frac{x^2}{Q(x)}$ as a sum of partial fractions. 4

c. If Q(x) is a factor of P(x), find the value of a and b. 4

Solution to the question no. 10

a Given,

$Q(x) = x^2 - 2x - 8$

$= x^2 - 4x + 2x - 8$

$= x(x - 4) + 2(x - 4)$

$= (x - 4)(x + 2)$ **(Ans.)**

b $\frac{x^2}{Q(x)} = \frac{x^2}{x^2 - 2x - 8}$

$$= \frac{(x^2 - 2x - 8) + (2x + 8)}{x^2 - 2x - 8}$$

$$= \frac{x^2 - 2x - 8}{x^2 - 2x - 8} + \frac{2x + 8}{x^2 - 2x - 8}$$

$= 1 + \frac{2x + 8}{(x - 4)(x + 2)}$ [Obtained from 'a']

$\therefore \frac{x^2}{Q(x)} = 1 + \frac{2x + 8}{(x - 4)(x + 2)}$ .... (i)

Suppose, $\frac{2x + 8}{(x - 4)(x + 2)} = \frac{A}{(x - 4)} + \frac{B}{(x + 2)}$ .... (ii)

Multiplying both sides of (ii) with (x – 4)(x + 2) we get,
$2x + 8 = A(x + 2) + B(x - 4)$ .... (iii)
Putting x = 4 in (iii) we get,
$2.4 + 8 = A(4 + 2) + B(4 - 4)$
Or, $8 + 8 = A.6 + B.0$
Or, $6A = 16$ or, $A = \frac{16}{6}$
$\therefore A = \frac{8}{3}$
Again putting x = –2 in (iii) we get,
$2(-2) + 8 = A(-2 + 2) + B(-2 - 4)$
Or, $-4 + 8 = A.0 + B(-6)$
Or, $4 = -6B$
Or, $B = \frac{-4}{6}$
$\therefore B = \frac{-2}{3}$
Putting the values of A and B in (i) we get,
$$\frac{2x+8}{(x-4)(x+2)} = \frac{\frac{8}{3}}{x-4} + \frac{\frac{-2}{3}}{x+2}$$
$$= \frac{8}{3(x-4)} - \frac{2}{3(x+2)}$$
From (i) we get,
$$\frac{x^2}{Q(x)} = 1 + \frac{8}{3(x-4)} - \frac{2}{3(x+2)} \text{ (Ans.)}$$

c Obtained from (a) we get,
$Q(x) = (x - 4)(x + 2)$
Given, $P(x) = x^3 - x^2 + ax + b$
If (x – 4) is a factor of P(x) then P(4) = 0
$\therefore P(4) = 4^3 - 4^2 + a(4) + b$
$= 64 - 16 + 4a + b$
$= 4a + b + 48$
According to the question, $4a + b + 48 = 0$ .... (iv)
Again, (x + 2) is a factor of P(x) then P(– 2) = 0
$\therefore P(-2) = (-2)^3 - (-2)^2 + a(-2) + b$
$= -8 - 4 - 2a + b = -2a + b - 12$
According to the question $-2a + b - 12 = 0$ .... (v)
Subtracting (v) from (iv) we get,
$4a + b + 48 + 2a - b + 12 = 0$
Or, $6a + 60 = 0$
Or, $6a = -60$
$\therefore a = -10$
Putting the value of a in (v) we get,
$-2(-10) + b - 12 = 0$
Or, $20 + b - 12 = 0$
Or, $b + 8 = 0$
$\therefore b = -8$
∵ The required value a = –10 and b = – 8. **(Ans.)**

Question ▶ 11 $f(x) = \frac{2x+2}{x-1}$ is a function, where $x \neq 1$.
[B.B. 16]

a. If $f(p) = k$, express p in terms of k. 2
b. Find $f^{-1}(3)$. 4
c. Express $f(x^2)$ as the sum of partial fractions. 4

Solution to the question no. 11

a Given,
$f(x) = \frac{2x+2}{x-1}$; where $x \neq 1$
$\therefore f(p) = \frac{2p+2}{p-1}$
According to the question, f(p) = k
Or, $\frac{2p+2}{p-1} = k$
Or, $2p + 2 = pk - k$
Or, $2p - pk = -k - 2$
Or, $p(2 - k) = -(k + 2)$
Or, $p = \frac{-(k+2)}{2-k}$
$\therefore p = \frac{k+2}{k-2}$ **(Ans.)**

b Suppose, $y = f(x) = \frac{2x+2}{x-1}$
$\therefore y = \frac{2x+2}{x-1}$
Or, $xy - y = 2x + 2$
Or, $xy - 2x = y + 2$
Or, $x(y - 2) = y + 2$
Or, $x = \frac{y+2}{y-2}$
Or, $f^{-1}(y) = \frac{y+2}{y-2}$ $\left[\because f(x) = y \; \therefore x = f^{-1}(y)\right]$
$\therefore f^{-1}(x) = \frac{x+2}{x-2}$
$\therefore f^{-1}(3) = \frac{3+2}{3-2} = \frac{5}{1} = 5$ **(Ans.)**

c Given,
$f(x) = \frac{2x+2}{x-1}$
$\therefore f(x^2) = \frac{2x^2+2}{x^2-1} = \frac{2(x^2+1)}{x^2-1}$
$= \frac{2(x^2-1+2)}{x^2-1}$
$= \frac{2(x^2-1)}{x^2-1} + \frac{4}{x^2-1}$
$\therefore f(x^2) = 2 + \frac{4}{(x+1)(x-1)}$ ....(i)
Suppose, $\frac{4}{(x+1)(x-1)} \equiv \frac{A}{x+1} + \frac{B}{x-1}$ ....(ii)
Multiplying both sides of (i) with (x + 1) (x – 1) we get,
$\therefore 4 \equiv A(x - 1) + B(x + 1)$..........(iii)
Putting x = 1 in (iii) we get, 4 = 2B ∴ B = 2
Putting x = –1 in (iii) we get, 4 = –2A ∴ A = –2
From (ii) we get,
$$\frac{4}{(x+1)(x-1)} = \frac{-2}{x+1} + \frac{2}{x-1}$$
Then from (i) $f(x^2) = 2 - \frac{2}{x+1} + \frac{2}{x-1}$ **(Ans.)**

Question ▶ 12 Given, P(x) = x.
[Mirzapur Cadet College, Tangail]

a. Find the leading coefficient and constant term of the expression $5x^6 - 3x^5 + 2x^2 - 7$. 2
b. Simplify: $\frac{1}{1+P(b)} + \frac{2}{1+P(b^2)} + \frac{4}{1+P(b^4)} + \frac{8}{1+P(b^8)} + \frac{16}{P(b^{16})-1}$. 4
c. Express $\frac{2P(x)}{1-P(x^4)}$ into partial fraction. 4

Solution to the question no. 12

a Given expression

$5x^6 - 3x^5 + 2x^2 - 7$ .... (i)

(i) is a expression with variable x and highest degree of x is 6.

$\therefore$ Leading co-efficient is 5 and constant term is -7 **(Ans.)**

b Given, $P(x) = x$

Now, $\frac{1}{1+P(b)} + \frac{2}{1+P(b^2)} + \frac{4}{1+P(b^4)} + \frac{8}{1+P(b^8)} + \frac{16}{P(b^{16})-1}$

$= \frac{1}{1+b} + \frac{2}{1+b^2} + \frac{4}{1+b^4} + \frac{8}{1+b^8} + \frac{16}{b^{16}-1}$

$= \frac{1}{1+b} + \frac{2}{1+b^2} + \frac{4}{1+b^4} + \frac{8}{1+b^8} + \frac{16}{(b^8+1)(b^8-1)}$

$= \frac{1}{1+b} + \frac{1}{1+b^2} + \frac{4}{1+b^4} + \left\{\frac{8(b^8-1)+16}{(b^8+1)(b^8-1)}\right\}$

$= \frac{1}{1+b} + \frac{2}{1+b^2} + \frac{4}{1+b^4} + \frac{8(b^8+1)}{(b^8+1)(b^8-1)}$

$= \frac{1}{1+b} + \frac{2}{1+b^2} + \frac{4}{1+b^4} + \frac{8}{b^8-1}$

$= \frac{1}{1+b} + \frac{2}{1+b^2} + \left\{\frac{4(b^4-1)+8}{(b^4+1)(b^4-1)}\right\}$

$= \frac{1}{1+b} + \frac{2}{1+b^2} + \frac{4(b^4+1)}{(b^4+1)(b^4-1)}$

$= \frac{1}{1+b} + \frac{2}{1+b^2} + \frac{4}{b^4-1}$

$= \frac{1}{1+b} + \left\{\frac{2(b^2-1)+4}{(b^2+1)(b^2-1)}\right\}$

$= \frac{1}{1+b} + \frac{2(b^2+1)}{(b^2-1)(b^2+1)}$

$= \frac{1}{1+b} + \frac{2}{b^2-1} = \frac{b-1+2}{(b+1)(b-1)}$

$= \frac{b+1}{(b+1)(b-1)} = \frac{1}{b-1}$

Ans. $\frac{1}{b-1}$

c Given, $P(x) = x$

$\therefore \frac{2P(x)}{1-P(x^4)}$

$= \frac{2x}{1-x^4}$

$= \frac{2x}{(1-x)(1+x)(1+x^2)}$

Let, $\frac{2x}{(1-x)(1+x)(1+x^2)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$ ......(i)

Multiplying both side by $(1-x)(1+x)(1+x^2)$, we get

$2x = A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1-x)(1+x)$

Or, $2x = A(1+x+x^2+x^3) + B(1-x+x^2-x^3) + Cx + D - Cx^3 - Dx^2$ ...... (ii)

Now, equating the co-efficient of x^3, x^2, x and constant term we get,

$A - B - C = 0$.................... (iii)

$A + B - D = 0$.................... (iv)

$A - B + C = 2$ .... (v)

$A + B + D = 0$ .... (vi)

From, (iii) + (v) we get,

$2(A - B) = 2$

Or, $A - B = 1$ .... (vii)

From, (iv) and (vi) we get,

$D = 0$

and $A + B = 0$ .... (viii)

From, (vii) and (viii) we get,

$A = \frac{1}{2}, B = -\frac{1}{2}$

From, (v) we get,

$C = 1$

Now, putting the value of A, B, C and D in equation (i) we get,

$\frac{2x}{(1-x)(1+x)(1+x^2)} = \frac{\frac{1}{2}}{1-x} + \frac{-\frac{1}{2}}{1+x} + \frac{x}{1+x^2}$ **(Ans.)**

Question ▸ 13 $x + 4$ is a factor of $P(x) = x^3 + 5x^2 + 6x + a$

[Mymensingh Girls' Cadet College, Mymensingh]

a. What is the value of a? 2

b. If $Q(x) = x^3 + 6x^2 + 7x + 10$. Find the common factor of P(x) and Q (x). 4

c If P(y) yields the same remainder upon division by y-m and y-n where $m \neq n$, show that $m^2 + n^2 + mn + 5m + 5n + 6 = 0$. 4

Solution to the question no. 13

a Given, $P(x) = x^3 + 5x^2 + 6x + a$

As (x+4) is a factor of P(x)

$\therefore P(-4) = 0$

Or, $(-4)^3 + 5(-4)^2 + 6(-4) + a = 0$

Or, $-64 + 80 - 24 + a = 0$

Or, $a - 8 = 0$

$\therefore$ $a = 8$ **(Ans.)**

b From 'a'

$a = 8$

$\therefore$ $P(x) = x^3 + 5x^2 + 6x + 8$

Given, $Q(x) = x^3 + 6x^2 + 7x + 10$

Now, (x + 4) is a factor P(x)

$P(x) = x^3 + 5x^2 + 6x + 8$

$= x^3 + 4x^2 + x^2 + 4x + 2x + 8$

$= x^2(x+4) + x(x+4) + 2(x+4)$

$= (x+4)(x^2+x+2)$

Again, $Q(x) = x^3 + 6x^2 + 7x + 10$

$= x^3 + 5x^2 + x^2 + 5x + 2x + 10$

$= x^2(x+5) + x(x+5) + 2(x+5)$

$= (x+5)(x^2+x+2)$

$\therefore$ Common factor for Q(x) and P(x) is $(x^2 + x + 2)$ **(Ans).**

c Similar to example – 6 of chapter–2 of your textbook.

Question ▸ 14 $P(x) = -x^2 - 15x - 10x^3 + 9$ & $Q(x) = x^3 + x^2 - 6x$. *[Pabna Cadet College, Pabna]*

a. Write P(x) as the ideal expression of x & find the leading co-efficient? 2

b. Resolve into Factors P(x)? 4

c. Express $\frac{x^2+x-1}{Q(x)}$ in partial fraction? 4

Solution to the question no. 14

a Given,

$P(x) = -x^2 - 15x - 10x^3 + 9$

In standard form, $P(x) = -10x^3 - x^2 - 15x + 9$

Here, degree of the polynomial is 3 and leading co-efficient is -10 **(Ans.)**

b From 'a'

$P(x) = -10x^3 - x^2 - 15x + 9$

$= -10x^3 + 5x^2 - 6x^2 + 3x - 18x + 9$

$= -5x^2(2x-1) - 3x(2x-1) - 9(2x-1)$

$= (2x-1)(-5x^2 - 3x - 9)$

$= -(2x-1)(5x^2 + 3x + 9)$

c Given that, $Q(x) = x^3 + x^2 - 6x$

$\therefore \dfrac{x^2+x-1}{Q(x)} = \dfrac{x^2+x-1}{x^3+x^2-6x} = \dfrac{x^2+x-1}{x^3+3x^2-2x^2-6x}$

$= \dfrac{x^2+x-1}{x^2(x+3)-2x(x+3)} = \dfrac{x^2+x-1}{(x+3)\,x(x-2)}$

$= \dfrac{x^2+x-1}{x(x+3)(x-2)}$

Let, $\dfrac{x^2+x-1}{x(x+3)(x-2)} \equiv \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$

$\therefore x^2 + x - 1 = A(x+3)(x-2) + B.x(x-2) + C.x(x+3)$

For $x = 0$,

$-1 = A.3(-2)$

Or, $A = \dfrac{1}{6}$

For $x = 2$

$4 + 2 - 1 = C.2.5$

Or, $5 = C.2.5$

Or, $C = \dfrac{1}{2}$

For, $x = -3$

$9 - 3 - 1 = B(-3)(-5)$

Or, $B = \dfrac{5}{15} = \dfrac{1}{3}$

$\therefore \dfrac{x^2+x-1}{x(x+3)(x-2)} = \dfrac{1}{6x} + \dfrac{1}{3(x+3)} + \dfrac{1}{2(x-2)}$ **(Ans.)**

Question ▸ 15 $F(a) = \dfrac{1}{1-a^3}$ and $P(x, y, z) = (x + y + z)(xy + yz + zx)$. *[Cumilla Cadet College, Cumilla]*

a. Verify whether $P(x, y, z)$ is cyclic or symmetric. 2

b. Express $F(a)$ into a partial fraction. 4

c. If $P(x, y, z) = xyz$, show that $\dfrac{1}{(x+y+z)^7} = \dfrac{1}{x^7} + \dfrac{1}{y^7} + \dfrac{1}{z^7}$. 4

Solution to the question no. 15

a Given that, $P(x, y, z) = (x + y + z)(xy + yz + zx)$

$= x^2y + xyz + zx^2 + xy^2 + y^2z + xyz + xyz + yz^2 + z^2x$

$= x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 + 3xyz$

$= xy(x + y) + yz(y + z) + zx(z + x) + 3xyz$

So, the given expression is a cyclic expression of the variables *x, y, z* because replacing *y* by *x* , *z* by *y* and *x* by *z* , the expression remains the same.

Again the given expression is symmetric expression of the variables, *x, y, z* because the expression remains unchanged when any two of the variables are interchanged.

b Given expression, $\dfrac{1}{1-a^3} = \dfrac{1}{(1-a)(1+a+a^2)}$

Let, $\dfrac{1}{(1-a)(1+a+a^2)} \equiv \dfrac{A}{1-a} + \dfrac{Ba+C}{1+a+a^2}$ ....(i)

Multiplying both sides of (i) by $(1-a)(1+a+a^2)$ we get,

$1 \equiv A(1 + a + a^2) + (Ba + C)(1 - a)$ ....(ii)

Putting $a = 1$ in (ii), we get,

$1 = A(1 + 1 + 1)$

Or, $3A = 1$

$\therefore A = \dfrac{1}{3}$

Equating the co-efficients of a^2, a, we get,

$A - B = 0$

$A + B - C = 0$

Putting $A = \dfrac{1}{3}$ in $A - B = 0$, we get,

$B = \dfrac{1}{3}$

Putting $A = \dfrac{1}{3}$, $B = \dfrac{1}{3}$ in $A + B - C = 0$, we get,

$\dfrac{1}{3} + \dfrac{1}{3} - C = 0$

Or, $\dfrac{2}{3} = C \quad \therefore C = \dfrac{2}{3}$.

Putting the values of A, B and C (i), we get,

$\dfrac{1}{(1-a)(1+a+a^2)} = \dfrac{\frac{1}{3}}{1-a} + \dfrac{\frac{1}{3}a + \frac{2}{3}}{1+a+a^2}$

$= \dfrac{1}{3(1-a)} + \dfrac{\frac{a+2}{3}}{1+a+a^2} = \dfrac{1}{3(1-a)} + \dfrac{a+2}{3(1+a+a^2)}$

This is the required expression of the given fraction into the partial fraction.

Ans: $\dfrac{1}{3(1-a)} + \dfrac{a+2}{3(1+a+a^2)}$

c Given that, $P(x, y, z) = (x + y + z)(xy + yz + zx)$

Or, $xyz = x^2y + xy^2 + xyz + xyz + y^2z + xyz + yz^2 + zx^2 + z^2x$

Or, $x^2y + xy^2 + xyz + y^2z + z^2x + yz^2 + x^2z + xyz = 0$

Or, $xy(x + y) + yz(x + y) + z^2(x + y) + xz(x + y) = 0$

Or, $(x + y)(xy + yz + z^2 + xz) = 0$

Or, $(x + y)\{y(x + z) + z(z + x)\} = 0$

$\therefore \quad (x + y)(y + z)(z + x) = 0$

Either, $x + y = 0$ or, $y + z = 0$ or, $z + x = 0$

$\therefore x = -y \quad \therefore y = -z \quad \therefore z = -x$

Now, $\dfrac{1}{(x+y+z)^7} = \dfrac{1}{(x-z+z)^7} = \dfrac{1}{x^7} \quad [\because y = -z]$

Again, $\dfrac{1}{x^7} + \dfrac{1}{y^7} + \dfrac{1}{z^7} = \dfrac{1}{x^7} + \dfrac{1}{(-z)^7} + \dfrac{1}{z^7} \quad [\because y = -z]$

$= \dfrac{1}{x^7} - \dfrac{1}{z^7} + \dfrac{1}{z^7} = \dfrac{1}{x^7}$

$\therefore \quad \dfrac{1}{(x+y+z)^7} = \dfrac{1}{x^7} + \dfrac{1}{y^7} + \dfrac{1}{z^7}$ **(Shown)**

Question ▸ 16 $P(a, b, c) = (a + b + c)(ab + bc + ca)$

$Q(a, b, c) = (b + c)(c + a)(a + b)$ *[Jhenidah Cadet College, Jhenidah]*

a. If $(x - y)^2 + (y - z)^2 = 0$, show that $x = y = z$ 2

b. Show that, $Q(a, b, c) + abc = P(a, b, c)$ 4

c. If $P(a, b, c) = abc$, justify $\dfrac{1}{(a+b+c)^9} = \dfrac{1}{a^9} + \dfrac{1}{b^9} + \dfrac{1}{c^9}$ is possible or not. 4

Solution to the question no. 16

a Given $(x - y)^2 + (y - z)^2 = 0$

we know, the sum of squares of two or more expressions will be zero if and only if their individual values will be zero.

So, $(x - y)^2 = 0$ and $(y -)^2 = 6$

Or, $x - y = 0$ or, $y - z = 0$

Or, $x = y$ or $y = z$

$\therefore \quad x = y = z$ **(Shown)**

b Here,

$Q(a, b, c) = (b + c)(c + a)(a + b)$

and $p(a, b, c) = (a + b + c)(ab + bc + ca)$

$\therefore Q(a, b, c) + abc = (b + c)(c + a)(a + b) + abc$

Considering the expression as a polynomial F(a) of a, we substitute $-b-c$ for a and get,

$F\{-(b-c)\} = (b + c)(c - b - c)(-b - c + b) + (-b - c)\, bc(b + c) - bc, (b + c) = 0.$

So, according to the factor theorem, $(a + b + c)$ is the factor of the the given expression.

The given expression having a cyclic homogeneous polynomial of degree 3 and one factor of degree 1 have been found. So, the remaining factor will be cyclic homogenous polynomial of degree two, i.e, will be of the form $k(a^2 + b^2 + c^2) + m(bc + ca + ab)$, there k and m are constants.

$\therefore (b + c)(c + a)(a + b) + abc = (a + b + c)\{k(a^2 + b^2 + c^2)\, m(bc + ca + ab)\}$ .... (i) For all values of a, b, c, (i) is true.

Putting in (i) first $a = 0, b = 0, c = 1$ and then $a = 1, b = 1, c = 0$, we get respectively $0 = k$ and $2 = 2(k \times 2 + m)$

$\therefore k = 0, m = 1.$

Now putting the values of k and m, we get, $(ab + c)(c + a)(a + b) + abc = (a + b + c)(bc + ca + ab).$

$\therefore$ $Q(a, b, c) + abc = (a + b + c)(ab + bc + ca)$ **(Shown)**

c Given that, $P(a, b, c) = (a + b + c)(ab + ca)$

Or, $abc = a^2b + ab^2 + abc + abc + b^2c + abc + bc^2 + ca^2 + c^2a$

Or, $a^2b + ab^2 + abc + b^2c + c^2a + bc^2 + a^2c + abc = 0$

Or, $ab(a + b) + bc(a + b) + c^2(a + b) + ac(a + b) - 0$

Or, $(a + b)(ab + bc + c^2 + ac) = 0$

Or, $(a + b)(b(a + c) + c(c + a)\} - 0$

$\therefore$ $(a + b)(b + c)(c + a) = 0$

Either, $a + b = 0$ Or, $b + c = 0$ Or, $c + a = 0$

$\therefore a = -b$ $\therefore b = -c$ $\therefore c = -a$

Now, $\frac{1}{(a+b+c)^2} = \frac{1}{(a-c+c)^9} = \frac{1}{a^9}$ $[\because b = -c]$

Again, $\frac{1}{a^9} + \frac{1}{b^9} + \frac{1}{c^9} = \frac{1}{a^9} + \frac{1}{(-c)^3} + \frac{1}{c^9}$ $[\because b = -c]$

$= \frac{1}{a^9} - \frac{1}{c^9} + \frac{1}{c^9}$

$= \frac{1}{a^9}$

$\therefore \frac{1}{(a+b+c)^9} = \frac{1}{a^9} + \frac{1}{b^9} + \frac{1}{c^9}$

Question ▶ 17 $p^a = q^b = r^c$, where $p \neq q \neq r$

[Jhenidah Cadet College, Jhenidah]

a. Show that, $\log_a \log_a \log_a(a^{a^{a^b}}) = b$ 2

b. If p, q, r are orderly proportional, prove that, $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ 4

c. If $pqr = 1$, show that, $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{3}{abc}$ 4

Solution to the question no. 17

a **Solution:** L.S. $= \log_a \log_a \log_a \left(a^{a^{a^b}}\right)$

$= \log_a \log_a a^{a^b} \log_a a$ $[\because \log_a P^r = r\log_a P]$

$= \log_a \log_a a^{a^b}.1$ $[\because \log_a a^1 = 1]$

$= \log_a a^b \log_a a$

$= \log_a a^b$ $[\because \log_a a = 1]$

$= b \log_a a = b \cdot 1 = b$

= R.S. **(Shown)**

b See chapter. 9.1, example-11 of your text book.

c Let,

$p^a = q^b = r^c = k$ [Where k is constant]

$\therefore p = k^{\frac{1}{a}};\ q = k^{\frac{1}{b}};\ r = k^{\frac{1}{c}}$

Given, $pqr = 1$

Or, $k^{\frac{1}{a}} . k^{\frac{1}{b}} . k^{\frac{1}{c}} = 1$

Or, $k^{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = k^0$

Or, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

$\therefore$ $\frac{1}{a} + \frac{1}{b} = -\frac{1}{c}$

Or, $\left(\frac{1}{a} + \frac{1}{b}\right)^3 = \left(-\frac{1}{c}\right)^3$ [cubing both sides]

Or, $\frac{1}{a^3} + \frac{1}{b^3} + 3\left(\frac{1}{a} + \frac{1}{b}\right)\frac{1}{a}.\frac{1}{b} = -\frac{1}{c^3}$

Or, $\frac{1}{a^3} + \frac{1}{b^3} + 3\left(\frac{1}{-c}\right)\frac{1}{a}.\frac{1}{b} = -\frac{1}{c^3}$

Or, $\frac{1}{a^3} + \frac{1}{b^3} - \frac{3}{abc} = -\frac{1}{c^3}$

$\therefore \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{3}{abc}$ **(Shown)**

Question ▶ 18 $f(x) = 18x^3 + 15x^2 - x + a$, $g(x) = x^3 + x^2 - 6x$ and $p = \frac{bc(a+d)}{(a-b)(a-c)} + \frac{ca(b+d)}{(b-c)(b-a)} + \frac{ab(c+d)}{(c-a)(c-b)}$ are three algebraic expressions. *[RAJUK Uttara Model College, Dhaka]*

a. Resolve into factors of g(x). 2

b. Find the value of 'a' if $(3x + 2)$ is a factor of the polynomial f(x). 4

c. Simplify p. 4

Solution to the question no. 18

a Given,

$g(x) = x^3 + x^2 - 6x$

$= x(x^2 + x - 6)$

$= x(x^2 + 3x - 2x - 6)$

$= x\{x(x + 3) - 2(x + 3)\}$

$= x(x + 3)(x - 2)$ **(Ans.)**

b Given, $f(x) = 18x^3 + 15x^2 - x + a$

Since $(3x + 2)$ is a factor of f(x). So if f(x) is divided by $(3x + 2)$ the remainder will be zero. That is-

$f\left(-\frac{2}{3}\right) = 0$

Or, $18\left(-\frac{2}{3}\right)^3 + 15\left(-\frac{2}{3}\right)^2 - \left(-\frac{2}{3}\right) + a = 0$

Or, $-18 . \frac{8}{27} + 15 . \frac{4}{9} + \frac{2}{3} + a = 0$

Or, $-\frac{16}{3} + \frac{20}{3} + \frac{2}{3} + a = 0$

Or, $a = \frac{16}{3} - \frac{20}{3} - \frac{2}{3}$

Or, $a = \frac{16 - 20 - 2}{3}$

Or, $a = \frac{-6}{3}$

$\therefore a = -2$ **(Ans.)**

c Given expression,

$$= \frac{bc(a+d)}{(a-b)(a-c)} + \frac{ca(b+d)}{(b-c)(b-a)} + \frac{ab(c+d)}{(c-a)(c-b)}$$

$$= \frac{bc(a+d)}{-(a-b)(c-a)} + \frac{ca(b+d)}{-(b-c)(a-b)} + \frac{ab(c+d)}{-(c-a)(b-c)}$$

$$= \frac{bc(a+d)(b-c)+ca(b+d)(c-a)+ab(c+d)(a-b)}{-(a-b)(b-c)(c-a)}$$

$$= \frac{abc(b-c)+abc(c-a)+abc(a-b)+d\{bc(b-c)+ca(c-a)+ab(a-b)\}}{-(a-b)(b-c)(c-a)}$$

But its numeratOr, $abc(b-c) + abc(c-a) + abc(a-b) = 0$

and $bc(b-c) + ca(c-a) + ab(a-b) = -(a-b)(b-c)(c-a)$

$\therefore$ Given expression $= \dfrac{0+d\{-(a-b)(b-c)(c-a)\}}{-(a-b)(b-c)(c-a)}$

$$= \frac{-d(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)} = d \textbf{ (Ans.)}$$

Question ▶ 19 $a^x = b^y = c^z$, where $a \neq b \neq c$, $M = \dfrac{p^{\frac{3}{2}}+pq}{pq-q^3} - \dfrac{\sqrt{p}}{\sqrt{p}-q}$.

[RAJUK Uttara Model College, Dhaka]

a. If $ab = c^2$ then, prove that, $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{2}{z}$. 2

b. If $abc = 1$ then, prove that, $\dfrac{1}{x^3}+\dfrac{1}{y^3}+\dfrac{1}{z^3}=\dfrac{3}{xyz}$. 4

c. Show that, $M = \dfrac{\sqrt{p}}{q}$. 4

Solution to the question no. 19

a See example-11 of ch-9.1 in Higher Math text book.

b Let, $a^x = b^y = c^z = k$ [Where k is constant]

$\therefore \quad a = k^{\frac{1}{x}}$

$b = k^{\frac{1}{y}}$

$c = k^{\frac{1}{z}}$

Given, $abc = 1$

Or, $k^{\frac{1}{x}} . k^{\frac{1}{y}} . k^{\frac{1}{z}} = 1$

Or, $k^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = k^0$ [$\because k^0 = 1$]

Or, $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$

$\therefore \quad \dfrac{1}{x}+\dfrac{1}{y}=-\dfrac{1}{z}$

Cubing both sides,

$$\left(\frac{1}{x}+\frac{1}{y}\right)^3 = \left(-\frac{1}{z}\right)^3$$

Or, $\dfrac{1}{x^3}+\dfrac{1}{y^3}+3\left(\dfrac{1}{x}+\dfrac{1}{y}\right)\dfrac{1}{x}\cdot\dfrac{1}{y} = -\dfrac{1}{z^3}$

Or, $\dfrac{1}{x^3}+\dfrac{1}{y^3}+3\left(\dfrac{1}{-z}\right)\dfrac{1}{x}\cdot\dfrac{1}{y} = -\dfrac{1}{z^3}$

Or, $\dfrac{1}{x^3}+\dfrac{1}{y^3}-\dfrac{3}{xyz} = -\dfrac{1}{z^3}$

$\therefore \dfrac{1}{x^3}+\dfrac{1}{y^3}+\dfrac{1}{z^3}=\dfrac{3}{xyz}$ **(Proved)**

c $M = \dfrac{p^{\frac{3}{2}}+pq}{pq-q^3} - \dfrac{\sqrt{p}}{\sqrt{p}-q}$

$= \dfrac{p(\sqrt{p}+q)}{q(p-q^2)} - \dfrac{\sqrt{p}}{\sqrt{p}-q}$ $\left[\because p^{\frac{3}{2}} = p \,.\, p^{\frac{1}{2}} = p\sqrt{p}\right]$

$= \dfrac{p(\sqrt{p}+q)}{q\{(\sqrt{p})^2-(q)^2\}} - \dfrac{\sqrt{p}}{\sqrt{p}-q}$

$= \dfrac{p(\sqrt{p}+q)}{q(\sqrt{p}+q)(\sqrt{p}-q)} - \dfrac{\sqrt{p}}{\sqrt{p}-q}$

$= \dfrac{p}{q(\sqrt{p}-q)} - \dfrac{\sqrt{p}}{\sqrt{p}-q} = \dfrac{p-q\sqrt{p}}{q(\sqrt{p}-q)}$

$= \dfrac{\sqrt{p}(\sqrt{p}-q)}{q(\sqrt{p}-q)}$ $[\because p = p^{\frac{1}{2}} . p^{\frac{1}{2}} = \sqrt{p} . \sqrt{p}]$

$= \dfrac{\sqrt{p}}{q}$ **(Ans.)**

Question ▶ 20 $f(x) = 18x^3 + 15x^2 - x + c$, $g(x) = x^2 - 4x - 7$ and $h(x) = x^3 - x^2 - 10x - 8$ are three polynomials of variable x.

[Dhaka Residential Model School and College, Dhaka]

a. Resolve h(x) into factors. 2

b. If $(3x + 2)$ is a factor of $f(x)$. Find the value of c. 4

c. Express $\dfrac{g(x)}{h(x)}$ as partial fractions. 4

Solution to the question no. 20

a Given, $h(x) = x^3 - x^2 - 10x - 8$

If $x = -1$ then $h(-1) = 0$.

Therefore $(x + 1)$ is a factor of h(x).

$\therefore h(x) = x^3 - x^2 - 10x - 8$

$= x^3 + x^2 - 2x^2 - 2x - 8x - 8$

$= x^2(x + 1) - 2x(x + 1) - 8(x + 1)$

$= (x + 1)(x^2 - 2x - 8)$

$= (x + 1)(x^2 - 4x + 2x - 8)$

$= (x + 1)\{x(x - 4) + 2(x - 4)\}$

$= (x + 1)(x + 2)(x - 4)$ **(Ans.)**

b Given, $f(x) = 18x^3 + 15x^2 - x + c$

Since $(3x + 2)$ is a factor of $f(x)$, according to inverse theorem of factor theorem, $f\left(-\dfrac{2}{3}\right) = 0$

Here, $f\left(-\dfrac{2}{3}\right) = 18\left(-\dfrac{2}{3}\right)^3 + 15\left(-\dfrac{2}{3}\right)^2 - \left(-\dfrac{2}{3}\right) + c$

$= -18.\dfrac{8}{27} + 15.\dfrac{4}{9} + \dfrac{2}{3} + c$

$= -\dfrac{16}{3} + \dfrac{20}{3} + \dfrac{2}{3} + c$

$= \dfrac{-16+20+2+3c}{3} = \dfrac{6+3c}{3}$

According to condition, $f\left(-\dfrac{2}{3}\right) = 0$

Or, $\dfrac{6+3c}{3} = 0$

Or, $6 + 3c = 0$

$\therefore \quad c = -2$ **(Ans.)**

c Given, $g(x) = x^2 - 4x - 7$

From 'a' we get,

$h(x) = (x + 1)(x + 2)(x - 4)$

$\therefore \frac{g(x)}{h(x)} = \frac{x^2 - 4x - 7}{(x + 1)(x + 2)(x - 4)}$

$\therefore \frac{x^2 - 4x - 7}{(x + 1)(x + 2)(x - 4)}$ is a proper fraction.

Let, $\frac{x^2 - 4x - 7}{(x + 1)(x + 2)(x - 4)} \equiv \frac{A}{x + 1} + \frac{B}{x + 2} + \frac{C}{x - 4}$ ..... (i)

Multiplying both sides of (i) by $(x + 1)(x + 2)(x - 4)$ we get,

$x^2 - 4x - 7 \equiv A(x + 2)(x - 4) + B(x + 1)(x - 4) + C(x + 1)(x + 2)$ .... (ii)

The equation (ii) is true for all values of x.

Putting $x = -1$ in equation (ii) we get,

$1 + 4 - 7 = A(-1 + 2)(-1 - 4)$

Or, $-2 = A(-5)$ $\therefore$ $A = \frac{2}{5}$

Putting $x = -2$ in equation (ii) we get,

$4 + 8 - 7 = B(-2 + 1)(-2 - 4)$

Or, $5 = B(-1)(-6)$ $\therefore$ $B = \frac{5}{6}$

Putting $x = 4$ in equation (ii) we get,

$16 - 16 - 7 = C(4 + 1)(4 + 2)$

Or, $-7 = C(5)(6)$ $\therefore$ $C = -\frac{7}{30}$

Now putting the values of A, B, C in equation (i) we get,

$$\frac{x^2 - 4x - 7}{(x + 1)(x + 2)(x - 4)} \equiv \frac{2}{5(x + 1)} + \frac{5}{6(x + 2)} - \frac{7}{30(x - 4)}$$

This is the expression of the given fraction into the partial fraction.

Question ▶ 21 $f(x) = x^3 + 5x^2 + 6x + 8$ and $g(x) = x^3 + x^2 - 6x$ are two algebraic expressions. *[Milestone College, Dhaka]*

a. Resolve g(x) into factors. 2

b. If $f(x)$ yields the same remainder upon division by $(x - m)$ and $(x - n)$ where $m \neq n$, then show that $m^2 + n^2 + mn + 5m + 5n + 6 = 0$ 4

c. Express $\frac{x^2 - 9x - 6}{g(x)}$ into partial fractions. 4

Solution to the question no. 21

a Given,

$g(x) = x^3 + x^2 - 6x$

$= x(x^2 + x - 6)$

$= x(x^2 + 3x - 2x - 6)$

$= x\{x(x + 3) - 2(x + 3)\}$

$= x(x + 3)(x - 2)$ **(Ans.)**

b Given,

$f(x) = x^3 + 5x^2 + 6x + 8$

If we divide f(x) by $(x - m)$ the remainder will be f(m).

$\therefore f(m) = m^3 + 5m^2 + 6m + 8$

Again, If we divide f(a) by $(x - n)$ the remainder will be f(n).

$\therefore f(n) = n^3 + 5n^2 + 6n + 8$

According to the question $f(m) = f(n)$

Or, $m^3 + 5m^2 + 6m + 8 = n^3 + 5n^2 + 6n + 8$

Or, $m^3 - n^3 + 5(m^2 - n^2) + 6(m - n) = 0$

Or, $(m - n)(m^2 + mn + n^2) + 5(m + n)(m - n) + 6(m - n) = 0$

Or, $(m - n)(m^2 + mn + n^2 + 5m + 5n + 6) = 0$

Since $m \neq n$, therefore, $m - n \neq 0$

$\therefore m^2 + mn + n^2 + 5m + 5n + 6 = 0$ **(Shown)**

c $\frac{x^2 - 9x - 6}{g(x)} = \frac{x^2 - 9x - 6}{x^3 + x^2 - 6x} = \frac{x^2 - 9x - 6}{x(x^2 + x - 6)}$

$= \frac{x^2 - 9x - 6}{x(x^2 + 3x - 2x - 6)}$

$\therefore \frac{x^2 - 9x - 6}{g(x)} = \frac{x^2 - 9x - 6}{x(x - 2)(x + 3)}$

Suppose, $\frac{x^2 - 9x - 6}{x(x - 2)(x + 3)} \equiv \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 3}$ .... (i)

Multiplying both sides by $x(x - 2)(x + 3)$ in (i) we get,

$x^2 - 9x - 6 \equiv A(x - 2)(x + 3) + Bx(x + 3) + Cx(x - 2)$ ..... (ii)

Which is true for all values of x.

Now putting the value of $x = 0$ is both sides of (ii) we get,

$0 - 0 - 6 = A(0 - 2)(0 + 3) + 0 + 0$

Or, $-6 = -6A$

$\therefore A = 1$

Again, putting the value of $x = 2$ both sides in (ii) we get,

$2^2 - 9.2 - 6 = 0 + B.2(2 + 3) + 0$

Or, $-20 = 10B$

$\therefore B = -2$

Putting the value of $x = -3$ in both sides of (ii) we get,

$(-3)^2 - 9.(-3) - 6 = 0 + 0 + C(-3)(-3 - 2)$

Or, $30 = 15C$

$\therefore C = 2$

Putting the value of A, B, C in (i) we get,

$$\frac{x^2 - 9x - 6}{x(x - 2)(x + 3)} = \frac{1}{x} - \frac{2}{x - 2} + \frac{2}{x + 3}$$ **(Ans.)**

Question ▶ 22 $P = \frac{x^2 - 9x - 6}{x(x - 2)(x + 3)}$, and $Q = a^{-3} + b^{-3} + c^{-3}$ are two equations. *[Bangladesh International School & College, Dhaka]*

a. Test whether Q is is cyclic or not. 2

b. If $Q = \frac{3}{abc}$, show that $bc + ca + ab = 0$ or $a = b = c$. 4

c. Express P as a sum of partial fractions. 4

Solution to the question no. 22

a Given $Q(a,b,c) = a^{-3} + b^{-3} + c^{-3}$

$= \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$

$\therefore Q(b, c, a) = \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{a^3}$

$= \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$

$= g(a, b, c)$

$\therefore$ Q is cycle expression **(Ans.)**

b Given, $Q = a^{-3} + b^{-3} + c^{-3}$

and $Q = \frac{3}{abc}$

$\therefore \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{3}{abc}$

then, $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} - \frac{3}{abc} = 0$

Or, $\left(\frac{1}{a}\right)^3 + \left(\frac{1}{b}\right)^3 + \left(\frac{1}{c}\right)^3 - 3.\frac{1}{a}.\frac{1}{b}.\frac{1}{c} = 0$

Or, $\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left\{\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2\right\} = 0$

$[\because x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\}]$

$\therefore$ Either, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Or, $\frac{bc+ca+ab}{abc}=0$

$\therefore$ bc + ca + ab = 0

Or, $\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{1}{c}-\frac{1}{a}\right)^2=0$

But, the sum of squares of two or more expressions will be zero if and only if their values will be zero individually.

So, $\left(\frac{1}{a}-\frac{1}{b}\right)^2=0$

Or, $\frac{1}{a}-\frac{1}{b}=0$

Or, $\frac{1}{a}=\frac{1}{b}$; [$\therefore$ a = b]

Again, $\left(\frac{1}{b}-\frac{1}{c}\right)^2=0$

Or, $\frac{1}{b}-\frac{1}{c}=0$

Or, $\frac{1}{b}=\frac{1}{c}$; [$\therefore$ b = c]

$\therefore$ a = b = c

Therefore if $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{3}{abc}$, then bc + ca + ab = 0

or, a = b = c **(Shown)**

c Suppose, $P=\frac{x^2-9x-6}{x(x-2)(x+3)}\equiv\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+3}$ ..... (i)

Multiplying both sides by x(x – 2) (x + 3) in (i) we get,

$x^2-9x-6\equiv A(x-2)(x+3)+Bx(x+3)+Cx(x-2)$ .... (ii)

Which is true for all values of x.

Now putting the value of x = 0 is both sides of (ii) we get,

$0-0-6=A(0-2)(0+3)+0+0$

Or, $-6=-6A$

$\therefore A=1$

Again, putting the value of x = 2 both sides in (ii) we get,

$2^2-9.2-6=0+B.2(2+3)+0$

Or, $-20=10B$

$\therefore B=-2$

Putting the value of x = – 3 in both sides of (ii) we get,

$(-3)^2-9.(-3)-6=0+0+C(-3)(-3-2)$

Or, $30=15C$

$\therefore C=2$

Putting the value of A, B, C in (i) we get,

$\frac{x^2-9x-6}{x(x-2)(x+3)}\equiv\frac{1}{x}-\frac{2}{x-2}+\frac{2}{x+3}$ **(Ans.)**

Question ▶ 23 $g(x)=\frac{1}{\sqrt{3x-1}}$, $p(x)=x^4+3x^3+4x^2+6x+4$, $f(a)=a^3+5a^2+6a+8$ *[Mirpur Girls' Ideal Laboratory Institute, Dhaka]*

a. Determine $g^{-1}(-1)$. 2

b. If f(a) yields the same remainder upon division by x – p and x – q where p ≠ q show that, $p^2+q^2+pq+5p+5q+6=0$ 4

c. Express $\frac{x}{p(x)}$ as partial fraction. 4

Solution to the question no. 23

a Give, $g(x)=\frac{1}{\sqrt{3x-1}}$

Let, $g(x)=y=\frac{1}{\sqrt{3x-1}}$

$\therefore x=g^{-1}(y)$

Now, $y=\frac{1}{\sqrt{3x-1}}$

Or, $\sqrt{3x-1}=\frac{1}{y}$

Or, $3x-1=\frac{1}{y^2}$

Or, $3x=\frac{1}{y^2}+1$

Or, $x=\frac{1+y^2}{3y^2}$

$\therefore g^{-1}(y)=\frac{1+y^2}{3y^2}$

$\therefore g^{-1}(-1)=\frac{1+(-1)^2}{3.(-1)^2}=\frac{2}{3}$ **(Ans.)**

b Similar to Example–6, Chapter–2 of your textbook. Page–47.

c Given, $p(x)=x^4+3x^3+4x^2+6x+4$

$=x^4+x^3+2x^3+2x^2+2x^2+2x+4x+4$

$=x^3(x+1)+2x^2(x+1)+2x(x+1)+4(x+1)$

$=(x+1)(x^3+2x^2+2x+4)$

$=(x+1)\{x^2(x+2)+2(x+2)\}$

$=(x+1)(x+2)(x^2+2)$

$\therefore \frac{x}{P(x)}=\frac{x}{(x+1)(x+2)(x^2+2)}$

Let, $\frac{x}{(x+1)(x+2)(x^2+2)}\equiv\frac{A}{x+1}+\frac{B}{x+2}+\frac{Cx+D}{x^2+2}$ ..... (i)

Multiplying both sides of (i) by $(x+1)(x+2)(x^2+2)$,

$x=A(x+2)(x^2+2)+B(x+1)(x^2+2)+(Cx+D)(x+1)(x+2)$ .... (ii)

Which is true for all values of x,

Now, putting x = – 1 in (ii),

$-1=A\{(-1)+2\}\{(-1)^2+2\}$

Or, $-1=A(1.3)$

Or, $-1=3A$

$\therefore A=-\frac{1}{3}$

Putting x = – 2 in (ii)

$-2=B\{(-2)+1\}\{(-2)^2+2\}$

Or, $-2=B(-1)\times 6$

Or, $-2=-6B$

Or, $B=\frac{1}{3}$

Equating the co-efficients of x^3 from (ii),

$A+B+C=0$

Or, $-\frac{1}{3}+\frac{1}{3}+C=0$

$\therefore C=0$

and $2A+B+3C+D=0$

Or, $2\left(-\frac{1}{3}\right)+\frac{1}{3}+3.0+D=0$

Or, $-\frac{2}{3}+\frac{1}{3}+D=0$

$\therefore D=\frac{1}{3}$

Putting values of A, B, C, D in (i)

$\frac{x}{(x+1)(x+2)(x^2+2)}=-\frac{1}{3(x+1)}+\frac{1}{3(x+2)}+\frac{1}{3(x^2+2)}$

$=\frac{1}{3}\left(-\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x^2+2}\right)$

Which is the required partial fraction. **(Ans.)**

Question ▶ 24 If $f(m) = m^3 + 5m^2 + 6m + 8$ and $h(m) = \frac{2m}{(m+1)(m^2+1)^2}$ are two algebrical expressions.

[Baridhara Scholars' Institution (BSI), Dhaka]

a. Find the value of $f(-3)$. 2

b. If f(m) yields the same remainder upon division by $(m - p)$ and $(m - q)$ where $p \neq q$, show that $p^2 + q^2 + pq + 5p + 5q + 6 = 0$. 4

c. Express h(m) as a sum of partial fraction. 4

Solution to the question no. 24

a Given,

$f(m) = m^3 + 5m^2 + 6m + 8$

$\therefore f(-3) = (-3)^3 + 5.(-3)^2 + 6.(-3) + 8$

$= -27 + 45 - 18 + 8$

$= 53 - 45$

$= 8$ **(Ans.)**

b Given,

$f(m) = m^3 + 5m^2 + 6m + 8$

If we divide f(m) by $(m - p)$ the remainder will be f(p).

$\therefore f(p) = p^3 + 5p^2 + 6p + 8$

Again, If we divide f(m) by $(m - q)$ the remainder will be f(q).

$\therefore f(q) = q^3 + 5q^2 + 6q + 8$

According to the question, f(p) = f(q)

Or, $p^3 + 5p^2 + 6p + 8 = q^3 + 5q^2 + 6q + 8$

Or, $p^3 - q^3 + 5(p^2 - q^2) + 6(p - q) = 0$

Or, $(p - q)(p^2 + pq + q^2) + 5(p + q)(p - q) + 6(p - q) = 0$

Or, $(p - q)(p^2 + pq + q^2 + 5p + 5q + 6) = 0$

Since $p \neq q$, therefore $p - q \neq 0$

$\therefore p^2 + pq + q^2 + 5p + 5q + 6 = 0$ **(Shown)**

c Given, $h(m) = \frac{2m}{(m+1)(m^2+1)^2}$

$\frac{2m}{(m+1)(m^2+1)^2} \equiv \frac{A}{m+1} + \frac{Bm+C}{m^2+1} + \frac{Dm+E}{(m^2+1)^2}$ ..... (i)

Multiplying both sides by $(m + 1)(m^2 + 1)^2$ in equation (i) we get,

$2m \equiv A(m^2 + 1)^2 + (Bm + C)(m + 1)(m^2 + 1) + (Dm + E)(m + 1)$ ...... (ii)

Substituting the value of $m = -1$ in equation (ii) we get,

$2(-1) = A(1 + 1)^2$

Or, $-2 = 4A$

Or, $A = -\frac{2}{4}$

$\therefore A = -\frac{1}{2}$

From (ii)

$2m \equiv A(m^4 + 2m^2 + 1) + (Bm + C)(m^3 + m + m^2 + 1) + Dm^2 + Dm + Em + E$

Or, $2m \equiv Am^4 + 2Am^2 + A + Bm^4 + Bm^2 + Bm^3 + Bm + Cm^3 + Cm + Cm^2 + C + Dm^2 + Dm + Em + E$

$\therefore \quad 2m = (A + B)m^4 + (B + C)m^3 + (2A + B + C + D)m^2 + (B + C + D + E)m + (A + C + E)$

Equating the coefficients of m^4, m^3, m^2 and m we get,

$A + B = 0$

Or, $-\frac{1}{2} + B = 0 \; [\because A = -\frac{1}{2}]$

$\therefore B = \frac{1}{2}$

Again, $B + C = 0$

Or, $\frac{1}{2} + C = 0$

$\therefore C = -\frac{1}{2}$

Again, $2A + B + C + D = 0$

Or, $2.\left(-\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{2} + D = 0$

Or, $-1 + D = 0$

$\therefore D = 1$

Again, $B + C + D + E = 2$

Or, $\frac{1}{2} - \frac{1}{2} + 1 + E = 2$

Or, $E = 2 - 1$

$\therefore E = 1$

Now, substituting the values of A, B, C, D and E in (i), we get,

$$\frac{2m}{(m+1)(m^2+1)^2} = \frac{-\frac{1}{2}}{m+1} + \frac{\frac{1}{2}m - \frac{1}{2}}{m^2+1} + \frac{m+1}{(m^2+1)^2}$$

$$= -\frac{1}{2(m+1)} + \frac{m-1}{2(m^2+1)} + \frac{m+1}{(m^2+1)^2}$$

Which is the desired partial fractions.

Question ▶ 25 $(x - 1)$ is a factor of the polynomial $g(x) = px^3 + qx^2 + rx + s$ and all co-efficients of the polynomial are integers and $p \neq 0$, $s \neq 0$. another expression $h(x) = x^4 - 2x^3 + 2x^2 - 2x + a$

[Chetona Model Academy (CMA), Dhaka]

a. Show that $p + q + r + s = 0$ 2

b. If p = 1, q = 5, r = 6 and s = 8 and g(x) yields the same remainder upon division by $x - k$ and $x - l$ where $k \neq l$ then show that $k^2 + l^2 + kl + 5k + 5l + 6 = 0$ 4

c. Express $\frac{1}{h(x)}$ as partial fraction. 4

Solution to the question no. 25

a Given,

$g(x) = px^3 + qx^2 + rx + s$

$(x - 1)$ is a factor of g(x)

i.e. $g(1) = 0$

or, $p.(1)^3 + q.(1)^2 + r.1 + s = 0$

$\therefore p + q + r + s = 0$ **(Shown)**

b Here,

$g(x) = px^3 + qx^2 + rx + s$

and p = 1, q = 5, r = 6, s = 8

$\therefore g(x) = x^3 + 5x^2 + 6x + 8$

If we divide g(x) by $(x - k)$ and $(x - l)$ the remainder will be g(k) and g(l) respectively

$g(k) = g(l)$

Or, $k^3 + 5k^2 + 6k + 8 = l^3 + 5l^2 + 6l + 8$

Or, $k^3 - l^3 + 5(k^2 - l^2) + 6(k - l) = 0$

Or, $(k - l)(k^2 + kl + l^2) + 5(k + l)(k - l) + 6(k - l) = 0$

Or, $(k - l)(k^2 + kl + l^2 + 5k + 5l + 6) = 0$

Since $k \neq l$, therefore $(k - l) \neq 0$

$\therefore k^2 + l^2 + kl + 5k + 5l + 6 = 0$. **(Shown)**

c Given expression,

$h(x) = x^4 - 2x^3 + 2x^2 - 2x + 1$

$\therefore h(1) = 1^4 - 2.1^3 + 2.1^2 - 2.1 + 1$

$= 1 - 2 + 2 - 2 + 1$

$= 0$

$\therefore (x - 1)$ is a factor of h(x)

Now, $h(x) = x^4 - 2x^3 + 2x^2 - 2x + 1$

$= x^4 - x^3 - x^3 + x^2 + x^2 - x - x + 1$

$= x^3(x - 1) - x^2(x - 1) + x(x - 1) - 1(x - 1)$

$= (x - 1)(x^3 - x^2 + x - 1)$

$= (x - 1)\{x^2(x - 1) + 1(x - 1)\}$

$= (x - 1)^2(x^2 + 1)$

$\therefore \frac{1}{h(x)} = \frac{1}{(x-1)^2(x^2+1)}$

Let, $\frac{1}{(x-1)^2(x^2+1)} \equiv \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+1)}$ ..... (i)

Multiplying both sides of (i) with $(x-1)^2(x^2+1)$ we get,

$1 \equiv A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$ ..... (ii)

Which is true for all values of x.

Now, putting x = 1 we get,

$1 = B(1^2+1)$

Or, $2B = 1$

$\therefore B = \frac{1}{2}$

Putting $B = \frac{1}{2}$ in (ii) we get,

$1 = A(x-1)(x^2+1) + \frac{1}{2}(x^2+1) + (Cx+D)(x-1)^2$

Or, $1 = A(x^3+x-x^2-1) + \frac{1}{2}x^2 + \frac{1}{2} + (Cx+D)(x^2-2x+1)$

Or, $1 = A(x^3-x^2+x-1) + \frac{1}{2}x^2 + \frac{1}{2} + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$

Or, $1 = x^3(A+C) + x^2\left(-A+\frac{1}{2}-2C+D\right) + x(A+C-2D) + \left(-A+\frac{1}{2}+D\right)$ .... (iii)

Now equating the co-efficients of x^3, x^2, x on both sides we get—

$A + C = 0$ .... (iv)

$-A - 2C + D + \frac{1}{2} = 0$ .... (v)

$A + C - 2D = 0$ .... (vi)

$-A + \frac{1}{2} + D = 1$ .... (vii)

Now, $A + C - 2D = 0$

Or, $2D = A + C$

Or, $2D = 0$ [$\because A + C = 0$]

Or, $D = 0$

Putting D = 0 in (vii) we get—

$-A + \frac{1}{2} = 1$

Or, $-A = \frac{1}{2}$

Or, $A = -\frac{1}{2}$

Again, $A + C = 0$

$\therefore C = -A = \frac{1}{2}$

Therefore,

$\frac{1}{(x-1)^2(x^2+1)} = \frac{\frac{-1}{2}}{(x-1)} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{1}{2}x}{x^2+1}$ **(Ans.)**

Question ▶ 26 $f(a) = a^3 + 5a^2 + 6a + 8$ and $g(a) = \frac{1}{a(a^2+1)^2}$ are two algebraic equation. *[Rajshahi Cantonment Public School & College, Rajshahi]*

a. Determine the value of f(–3). 2

b. If f(x) yields the same remainder upon division by (x – p) and (x – q) where p ≠ q, show that $p^2 + q^2 + pq + 5p + 5q + 6 = 0$. 4

c. Express g(a) into partial fraction. 4

Solution to the question no. 26

a Given,

$f(a) = a^3 + 5a^2 + 6a + 8$

$\therefore f(-3) = (-3)^3 + 5.(-3)^2 + 6.(-3) + 8$

$= -27 + 45 - 18 + 8$

$= 53 - 45$

$= 8$ **(Ans.)**

b Given,

$f(a) = a^3 + 5a^2 + 6a + 8$

If we divide f(a) by (x – p) the remainder will be f(p).

$\therefore f(p) = p^3 + 5p^2 + 6p + 8$

Again, if we divide f(a) by (x – q) the remainder will be f(q).

$\therefore f(q) = q^3 + 5q^2 + 6q + 8$

According to the question f(p) = f(q)

Or, $p^3 + 5p^2 + 6p + 8 = q^3 + 5q^2 + 6q + 8$

Or, $p^3 - q^3 + 5(p^2 - q^2) + 6(p - q) = 0$

Or, $(p-q)(p^2+pq+q^2) + 5(p+q)(p-q) + 6(p-q) = 0$

Or, $(p-q)(p^2+pq+q^2+5p+5q+6) = 0$

Since p ≠ q, therefore, p – q ≠ 0

$\therefore p^2 + pq + q^2 + 5p + 5q + 6 = 0$ **(Shown)**

c See the text book example-7; Page-62.

Question ▶ 27 If $f(x) = x^3 + 5x^2 + 6x + 8$, and $g(x) = \frac{1}{x^2(x^2+1)^2}$ are two algebraic expressions.

[Millennium Scholastic School & College, Bogura]

a. Define : Algebraic expression. 2

b. If $f(x) = x^3 + 5x^2 + 6x + 8$, yields the same remainder upon division by (x–a) and (x–b), Where a ≠ b, show that, $a^2 + b^2 + ab + 5a + 5b + 6 = 0$ 4

c. Express g(x) into partial fraction. 4

Solution to the question no. 27

a **Algebraic expression :**

When one or more members and symbols representing numbers are combined meaningfully by any one or more signs, then a new symbol representing numbers is created. This symbol is called algebraic expression.

b Given,

$f(x) = x^3 + 5x^2 + 6x + 8$

If we divide f(x) by (x – a) the remainder will be f(a).

$\therefore f(a) = a^3 + 5a^2 + 6a + 8$

Again, If we divide f(a) by (x – b) the remainder will be f(b).

$\therefore f(b) = b^3 + 5b^2 + 6b + 8$

According to the question f(a) = f(b)

Or, $a^3 + 5a^2 + 6a + 8 = b^3 + 5b^2 + 6b + 8$

Or, $a^3 - b^3 + 5(a^2 - b^2) + 6(a - b) = 0$

Or, $(a-b)(a^2+ab+b^2) + 5(a+b)(a-b) + 6(a-b) = 0$

Or, $(a-b)(a^2+ab+b^2+5a+5b+6) = 0$

Since a ≠ b, therefore, a – b ≠ 0

$\therefore a^2 + ab + b^2 + 5a + 5b + 6 = 0$ **(Shown)**

c $\frac{1}{x^2(x^2+1)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$ — (i)

Multiplying both sides by $x^2(x+1)^2$ of (i), we get

$1 \equiv Ax(x^2+1)^2 + B(x^2+1)^2 + (Cx+D)(x^2(x^2+1) + (Ex+F)x^2$

Or, $1 \equiv Ax(x^4+2x^2+1) + B(x^4+2x^2+1) + (Cx+D)(x^4+x^2) + (Ex+F)x^2$

Or, $1 \equiv A(x^5+2x^3+x) + B(x^4+2x^2+1) + C(x^5+x^3) + D(x^4+x^2) + Ex^3 + Fx^2$

Or, $1 \equiv (A + C)x^5 + (B + D)x^4 + (2A + C + E)x^3 + (2B + D + F)\, x^2 + AX + B$ — (ii)

Equating the coefficients of x^5, x^4, x^3, x^2 x and constant of a (ii), we get

$A + C = 0$ — (a)

$B + D = 0$ — (b)

$2A + C + E = 0$ — (c)

$2B + D + F = 0$ — (d)

$A = 0$ — (e)

$B = 1$ — (f)

From (a) and (e), we get,

$A = 0$

$\therefore C = 0$

From (c),

$2.\,0 + E = 0$

$\therefore E = 0$

From (b) and (f),

$B = 1$

$\therefore B + D = 0$

Or, $1 + D = 0$

$\therefore D = -1$

From (d),

$2 \times 1 - 1 + F = 0$

Or, $1 + F = 0$

$\therefore F = -1$

Putting the values of A, B, C, D, E, F in (i)

we get $\frac{1}{x^2(x^2+1)^2} = \frac{0}{x} + \frac{1}{x^2} + \frac{0.x - 1}{x^2 + 1} + \frac{0.x - 1}{(x^2+1)^2}$

$\therefore \frac{1}{x^2(x^2+1)^2} = \frac{1}{x^2} - \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}$ **(Ans.)**

Question ▶ 28 $F(p, q, r) = p^{-3} + q^{-3} + r^{-3} - 3p^{-1}q^{-1}r^{-1}$ and $g(x) = \frac{x^2}{x^2 - 4}$ *[Cantonment Public School & College, Saidpur]*

a. Show that x + 1 is the factor of $4x^4 - 5x^3 + 5x - 4$. 2

b. If $f(p, q, r) = 0$, then show that $p = q = r$ or $pq + qr + rp = 0$. 4

c. Express as the sum of partial fractions of g(x). 4

Solution to the question no. 28

a Let, $f(x) = 4x^4 - 5x^3 + 5x - 4$

$\therefore f(-1) = 4(-1)^4 - 5(-1)^3 + 5(-1) - 4$

$= 4 + 5 - 5 - 4$

$= 9 - 9$

$= 0$

$\therefore$ (x + 1) is a factor of $4x^4 - 5x^3 + 5x - 4$ **(Shown)**

b If $f(p, q, r) = 0$

Then, $p^{-3} + q^{-3} + r^{-3} - 3p^{-1}q^{-1}r^{-1} = 0$

Or, $\frac{1}{p^3} + \frac{1}{q^3} + \frac{1}{r^3} - \frac{3}{pqr} = 0$

Or, $\left(\frac{1}{p}\right)^3 + \left(\frac{1}{q}\right)^3 + \left(\frac{1}{r}\right)^3 - 3\frac{1}{p}.\frac{1}{q}.\frac{1}{r} = 0$

Or, $\frac{1}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\left\{\left(\frac{1}{p} - \frac{1}{q}\right)^2 + \left(\frac{1}{q} - \frac{1}{r}\right)^2 + \left(\frac{1}{r} - \frac{1}{p}\right)^2\right\}$

[$\because x^3+y^3+z^3-3xyz = \frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}$]

$\therefore \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$

Or, $\frac{qr + rp + pq}{pqr} = 0$

$\therefore pq + qr + rp = 0$

Or, $\left(\frac{1}{p} - \frac{1}{q}\right)^2 + \left(\frac{1}{q} - \frac{1}{r}\right)^2 + \left(\frac{1}{r} - \frac{1}{p}\right)^2 = 0$

But the sum of squares of two or more expressions will be zero if and only if their values will be zero individually.

So, $\left(\frac{1}{p} - \frac{1}{q}\right)^2 = 0$ | Again, $\left(\frac{1}{q} - \frac{1}{r}\right)^2$

Or, $\frac{1}{p} - \frac{1}{q} = 0$ | Or, $\frac{1}{q} - \frac{1}{r} = 0$

Or, $\frac{q - p}{pq} = 0$ | Or, $\frac{r - q}{qr} = 0$

Or, $q - p = 0$ | Or, $r - q = 0$

$\therefore q = p$ | $\therefore r = q$

$\therefore p = q = r$.

Therefore, $p = q = r$ or $pq + qr + rp = 0$ **(Shown)**

c Given, $g(x) = \frac{x^2}{x^2 - 4}$

$= \frac{x^2}{(x + 2)(x - 2)}$

Let, $\frac{x^2}{(x + 2)(x - 2)} = 1 + \frac{A}{x + 2} + \frac{B}{x - 2}$ .... (i)

Multiplying both sides by (x + 2) (x – 2), we get

$x^2 = (x + 2)(x - 2) + A(x - 2) + B(x + 2)$ .... (ii)

Putting x = 2 in (ii) we get,

$2^2 = (2 + 2)(2 - 2) + A(2 - 2) + B(2 + 2)$

Or, $4 = 0 + 0 + 4B$

Or, $4B = 4$

$\therefore B = 1$

Again putting x = – 2 in (ii) we get,

$(-2)^2 = (-2 + 2)(-2 - 2) + A(-2 - 2) + B(-2 + 2)$

Or, $4 = 0 - 4A + 0$

Or, $4 = -4A$

$\therefore A = -1$

Now from (i), we get,

$\frac{x^2}{x^2 - 4} = 1 - \frac{1}{x + 2} + \frac{1}{x - 2}$ **(Ans.)**

Question ▶ 29 $f(x) = \frac{x + 3}{1 - 2x}$ is a function and $P(x) = x^3 - 2x^2 + 1$ is a polynomial with variable x. *[Cantonment English School & College, Chattogram]*

a. Find the domain of f(x). 2

b. Show that f(x) is one-one function. Also find the range of f(x). 4

c. Express $\frac{P(x)}{x^2 - 2x - 3}$ as partial fractions. 4

Solution to the question no. 29

a Given, $f(x) = \frac{x + 3}{1 - 2x}$

f(x) will be defined if and only if

$1 - 2x \neq 0$

Or, $2x \neq 1$

$\therefore x \neq \frac{1}{2}$

$\therefore$ Domain $= \mathbb{R} - \left\{\frac{1}{2}\right\}$ **(Ans.)**

b For any $x_1, x_2 \in$ domain f, f(x) will be one-one if and only if $f(x_1) = f(x_2)$, then $x_1 = x_2$

Let, $f(x_1) = f(x_2)$

Or, $\frac{x_1 + 3}{1 - 2x_1} = \frac{x_2 + 3}{1 - 2x_2}$

Or, $(x_1 + 3)(1 - 2x_2) = (1 - 2x_1)(x_2 + 3)$

Or, $x_1 - 2x_1x_2 + 3 - 6x_2 = x_2 - 2x_1x_2 + 3 - 6x_1$

Or, $x_1 + 6x_1 = x_2 + 6x_2 - 2x_1x_2 + 3 + 2x_1x_2 - 3$

Or, $7x_1 = 7x_2$

$\therefore x_1 = x_2$

$\therefore$ The function f(x) in one-one **(Shown)**

Let, y = f(x), then $x = f^{-1}(y)$

Or, $y = \frac{x + 3}{1 - 2x}$

Or, $y - 2xy = x + 3$

Or, $x + 2xy = y - 3$

Or, $x(1 + 2y) = y - 3$

$\therefore x = \frac{y - 3}{1 + 2y}$

If we put $y = -\frac{1}{2}$ then the function will be undefined.

$\therefore$ Range $= \mathbb{R} - \left\{-\frac{1}{2}\right\}$ **(Ans.)**

c The expression is $\frac{x^3 - 2x^2 + 1}{x^2 - 2x - 3}$

Now, $\frac{x^3 - 2x^2 + 1}{x^2 - 2x - 3} = \frac{x(x^2 - 2x - 3) + 3x + 1}{x^2 - 2x - 3} = x + \frac{3x + 1}{(x - 3)(x + 1)}$

Now, $\frac{3x + 1}{(x - 3)(x + 1)}$ is a proper fraction.

Let, $\frac{3x + 1}{(x - 3)(x + 1)} \equiv \frac{A}{x - 3} + \frac{B}{x + 1}$ ....(i)

Multiplying (x – 3) (x + 1) in both sides of (i)

$3x + 1 \equiv A(x + 1) + B(x - 3)$ ....(ii)

Putting $x = -1$ in both sides of (ii)

$3(-1) + 1 = A(-1 + 1) + B(-1 - 3)$

Or, $-3 + 1 = A.0 - 4B$

Or, $-2 = -4B$

$\therefore B = \frac{2}{4} = \frac{1}{2}$

Again, putting $x = 3$ to both sides of (ii)

$3.3 + 1 = A(3 + 1) + B(3 - 3)$

Or, $9 + 1 = 4.A + B.0$

Or, $10 = 4A$

$\therefore A = \frac{10}{4} = \frac{5}{2}$

Putting values of A & B in (i)

$\frac{3x + 1}{(x - 3)(x + 1)} = \frac{\frac{5}{2}}{x - 3} + \frac{\frac{1}{2}}{x + 1} = x + \frac{\frac{5}{2}}{x - 3} + \frac{\frac{1}{2}}{x + 1}$

$\therefore$ Required partial fraction

$\frac{x^3 - 2x^2 + 1}{x^2 - 2x - 3} = x + \frac{5}{2(x - 3)} + \frac{1}{2(x + 1)}$ **(Ans.)**

Question ▶ 30 $F(x) = \frac{3x}{x - 1}$ is a function.

[Bangladesh Mahila Somitee Girls' High School & College, Chattogram]

a. Determine general term of the sequence $1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}}$, ... 2

b. If $F : \mathbb{R} \rightarrow \mathbb{R}$, then determine whether the function F one-one and onto or not. 4

c. Express $F(x^4)$ as sum of partial fractions. 4

Solution to the question no. 30

a Here, the sequence,

$1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}}$, ...

$\therefore$ First term, a = 1

Common ratio, $r = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}$

$\therefore$ General or n^{th} term $= ar^{n-1}$

$= 1.\left(\frac{1}{\sqrt{2}}\right)^{n-1}$

$= \left(\frac{1}{\sqrt{2}}\right)^{n-1}$ **(Ans.)**

b Given, $F : \mathbb{R} \rightarrow \mathbb{R}$, $F(x) = \frac{3x}{x - 1}$

For $x_1, x_2 \in$ Dom F,

let $x_1, x_2 \in$ Dom F,

The function will be one one if and only if $F(x_1) = F(x_2)$ then $x_1 = x_2$

$\therefore F(x_1) = F(x_2)$

Or, $\frac{3x_1}{x_1 - 1} = \frac{3x_2}{x_2 - 1}$

Or, $3x_1x_2 - 3x_1 = 3x_1x_2 - 3x_2$

Or, $-3x_1 = -3x_2$

Or, $x_1 = x_2$

so, F(x) is one-one

Again, for any y $\in$ Range of F,

Let

$y = f(x)$

Or, $y = \frac{3x}{x - 1}$

Or, $xy - y = 3x$

Or, $xy - 3x = y$

Or, $x(y - 3) = y$

Or, $x = \frac{y}{y - 3}$

$\therefore f\left(\frac{y}{y - 3}\right) = \frac{3.\frac{y}{y - 3}}{\frac{y}{y - 3} - 1}$

$= \frac{\frac{3y}{y - 3}}{\frac{y - y + 3}{y - 3}}$

$= \frac{3y}{3}$

$= y = f(x)$

$\therefore$ f(x) is onto function

Therefore, f(x) is one-one and onto. **(Ans.)**

c Given, $F(x) = \frac{3x}{x - 1}$

$\therefore F(x^4) = \frac{3x^4}{x^4 - 1}$

$= \frac{3x^4 - 3 + 3}{x^4 - 1}$

$$= \frac{3(x^4-1)}{x^4-1} + \frac{3}{(x^2-1)(x^2+1)}$$

$$= 3 + \frac{3}{(x-1)(x+1)(x^2+1)}$$

Let, $\frac{3}{(x-1)(x+1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$ .... (i)

Multiplying both sides by $(x-1)(x+1)(x^2+1)$, we get,

$3 \equiv A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$

Or, $3 \equiv A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + Cx^3 + Dx^2 - Cx - D$ .... (ii)

Now, equating the co-efficient of x^3, x^2, x and constant term of (ii), we get,

$A + B + C = 0$ .... (a)

$A - B + D = 0$ .... (b)

$A + B - C = 0$ .... (c)

$A - B - D = 3$ .... (d)

From (a) and (c), we get, $C = 0$

From (b) and (d), we get, $2D = -3$

$$D = -\frac{3}{2}$$

Putting the value of C and D in (a) and (b), we get,

$$A + B = 0$$

$$A - B = \frac{3}{2}$$

$$2A = \frac{3}{2}$$

$$A = \frac{3}{4}$$

From (a), $B = -\frac{3}{4}$

Putting the value of A, B, C and D in equation (i), we get,

$$\frac{3}{(x-1)(x+1)(x^2+1)} = \frac{\frac{3}{4}}{x-1} + \frac{-\frac{3}{4}}{x+1} + \frac{-\frac{3}{2}}{x^2+1}$$

$\therefore \quad F(x^4) = 3 + \frac{3}{4(x-1)} - \frac{3}{4(x+1)} - \frac{3}{2(x^2+1)}$ **(Ans.)**

Question ▶ 31 $x - 1$ is a factor of the polynomial $g(x) = px^3 + qx^2 + rx + s$ and all coefficients of the polynomial are integers and $p \neq 0$, $s \neq 0$ another Expression $Q(x) = \frac{x^3}{x^2-16}$

[Navy Anchorage School and College, Chattogram]

a. Show that, $p + q + r + s = 0$ 2

b. If $p = 1$, $q = 5$, $r = 6$ and $s = 8$ and g(x) yields the same remainder upon division by $x - a$ and $x - b$ where $a \neq b$ then show that $a^2 + b^2 + ab + 5a + 5b + 6 = 0$ 4

c. Express Q(x) as partial fraction. 4

Solution to the question no. 31

a Given,

$g(x) = px^3 + qx^2 + rx + s$

$(x - 1)$ is a factor of g(x)

i.e. $g(1) = 0$

Or, $p.(1)^3 + q.(1)^2 + r.1 + s = 0$

$\therefore p + q + r + s = 0$ **(Shown)**

b Here,

$g(x) = px^3 + qx^2 + rx + s$

and $p = 1, q = 5, r = 6, s = 8$

$\therefore g(x) = x^3 + 5x^2 + 6x + 8$

If we divide g(x) by $(x - a)$ and $(x - b)$ the remainder will be g(a) and g(b) respectively

$g(a) = g(b)$

Or, $a^3 + 5a^2 + 6a + 8 = b^3 + 5b^2 + 6b + 8$

Or, $a^3 - b^3 + 5(a^2 - b^2) + 6(a - b) = 0$

Or, $(a - b)(a^2 + ab + b^2) + 5(a + b)(a - b) + 6(a - b) = 0$

Or, $(a - b)(a^2 + ab + b^2 + 5a + 5b + 6) = 0$

Since $a \neq b$, therefore $(a - b) \neq 0$

$\therefore a^2 + b^2 + ab + 5a + 5b + 6 = 0$. **(Shown)**

c Given,

$$Q(x) = \frac{x^3}{x^2-16}$$

$$= \frac{x(x^2-16)+16x}{x^2-16} = x + \frac{16x}{x^2-16}$$

$$= x + \frac{16x}{(x+4)(x-4)}$$

Suppose, $\frac{16x}{(x+4)(x-4)} \equiv \frac{A}{x+4} + \frac{B}{x-4}$ .... (i)

Multiplying both sides of (i) with $(x + 4)(x - 4)$ in (i) we get,

$16 = A(x-4) + B(x+4)$ .... (ii)

Which is true for all values of x.

Putting $x = -4$ both sides in (ii) we get,

$16 \times (-4) = A(-4-4) + B(-4+4)$

Or, $-64 = -8A$

$\therefore A = 8$

Again, putting $x = 4$ both sides in (ii) we get,

$16 \times 4 = A(4-4) + B(4+4)$

Or, $64 = 8B$

$\therefore B = 8$

Putting the value of A and B in (i) we get,

$$\frac{16x}{(x+4)(x-4)} = \frac{8}{x+4} + \frac{8}{x-4} = 8\left(\frac{1}{x+4} + \frac{1}{x-4}\right)$$

$\therefore Q(x) = x + 8\left(\frac{1}{x+4} + \frac{1}{x-4}\right)$;

This is the expression of the given fraction into partial fraction. **(Ans.)**

Question ▶ 32 Let $f(x) = x^3$ *[SCHOLARSHOME, Sylhet]*

a. Resolve into factor: $x^3 + 2x^2 - 5x - 6$ 2

b. If $\frac{1}{f(a)} + \frac{1}{f(b)} + \frac{1}{f(c)} = \frac{3}{abc}$, then prove that, $ab + bc + ca = 0$ or, $a = b = c$. 4

c. Express $\frac{f(x)}{f(x)-1}$ into partial fractions. 4

Solution to the question no. 32

a Let, $f(x) = x^3 + 2x^2 - 5x - 6$

The leading coefficient and constant term of $f(x)$ are 1 and -6

The set of factors of the constant term -6 of

$f(x) = \{1, -1, 2, -2, 3, -3, 6, -6\}$

$\therefore f(1) = 1^3 + 2.1^2 - 5.1 - 6 = -8 \neq 0$

$\therefore f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$
$= -1 + 2 + 5 - 6 = 0$

$\therefore \{x - (-1)\}$, that is $(x + 1)$ is a factor of $f(x)$.

Now, $x^3 + 2x^2 - 5x - 6$
$= x^3 + x^2 + x^2 + x - 6x - 6$
$= x^2(x + 1) + x(x + 1) - 6(x + 1)$
$= (x + 1)(x^2 + x - 6)$
$= (x + 1)(x - 2)(x + 3)$ **(Ans.)**

b Given, $f(x) = x^3$
$\therefore f(a) = a^3$
$f(b) = b^3$
$f(c) = c^3$

$\therefore \frac{1}{f(a)} + \frac{1}{f(b)} + \frac{1}{f(c)} = \frac{3}{abc}$

Or, $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} - \frac{3}{abc} = 0$

Or, $\left(\frac{1}{a}\right)^3 + \left(\frac{1}{b}\right)^3 + \left(\frac{1}{c}\right)^3 - 3.\frac{1}{a}.\frac{1}{b}.\frac{1}{c} = 0$

Or, $\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left\{\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2\right\} = 0$

$[\because x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\}]$

$\therefore$ Either $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

Or, $\frac{bc + ca + ab}{abc} = 0$

$\therefore bc + ca + ab = 0$

Or, $\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2 = 0$

But, the sum of squares of two or more expressions will be zero if and only if their values will be zero individually.

So, $\left(\frac{1}{a} - \frac{1}{b}\right)^2 = 0$

Or, $\frac{1}{a} - \frac{1}{b} = 0$

Or, $\frac{1}{a} = \frac{1}{b}$ $\therefore a = b$

Again, $\left(\frac{1}{b} - \frac{1}{c}\right)^2 = 0$

Or, $\frac{1}{b} - \frac{1}{c} = 0$

Or, $\frac{1}{b} = \frac{1}{c}$ $\therefore b = c$

$\therefore a = b = c$

Therefore, if $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{3}{abc}$, then $bc + ca + ab = 0$

Or, $a = b = c$ **(Shown)**

c Given, $f(x) = x^3$

$\therefore \frac{f(x)}{f(x) - 1} = \frac{x^3}{x^3 - 1} = \frac{x^3 - 1 + 1}{x^3 - 1}$

$= \frac{x^3 - 1}{x^3 - 1} + \frac{1}{x^3 - 1}$

$= 1 + \frac{1}{(x - 1)(x^2 + x + 1)}$

Let, $\frac{1}{(x - 1)(x^2 + x + 1)} \equiv \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$ .... (i)

Multiplying both sides of (i) by $(x - 1)(x^2 + x + 1)$

We get, $1 = A(x^2 + x + 1) + (Bx + C)(x - 1)$ .... (ii)

Putting $x = 1$ in (ii) we get,

$1 = A(1 + 1 + 1) + 0$

Or, $3A = 1$

$\therefore A = \frac{1}{3}$

Equating the co-efficients of x^2, x, we get,

$A + B = 0$

$A - B + C = 0$

Putting $A = \frac{1}{3}$ in $A + B = 0$,

We get,

$\frac{1}{3} + B = 0$

$\therefore B = -\frac{1}{3}$

Putting $A = \frac{1}{3}$ and $B = -\frac{1}{3}$ in $A - B + C = 0$,

We get,

$\frac{1}{3} - \left(-\frac{1}{3}\right) + C = 0$

Or, $\frac{1}{3} + \frac{1}{3} + C = 0$

Or, $\frac{2}{3} + C = 0$

$\therefore C = -\frac{2}{3}$

Putting the values of A, B and C in (i), we get,

$$\frac{1}{(x - 1)(x^2 + x + 1)} = \frac{\frac{1}{3}}{x - 1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2 + x + 1}$$

$$= \frac{1}{3(x - 1)} - \frac{\frac{x + 2}{3}}{x^2 + x + 1}$$

$$= \frac{1}{3(x - 1)} - \frac{x + 2}{3(x^2 + x + 1)}$$

$\therefore \frac{f(x)}{f(x) - 1} = 1 + \frac{1}{3(x - 1)} - \frac{x + 2}{3(x^2 + x + 1)}$ **(Ans.)**

Question ▶ 33 $f(x) = \frac{x + 1}{3x - 1}$ and $g(x) = \frac{x^2 + x - 1}{x^3 - 4x}$.

[Secondary & Higher Secondary Education Board, Jashore]

a. If $(x + 1)$ is a factor of $p(x) = x^2 - kx + 1$, find k. 2
b. Find the domain of $f^{-1}(x)$. 4
c. Express $g(x)$ in partial fraction. 4

Solution to the question no. 33

a Given,

$p(x) = x^2 - kx + 1$

Since $(x + 1)$ is a factor of $p(x)$

$p(-1) = 0$

Or, $(-1)^2 - k(-1) + 1 = 0$

Or, $1 + k + 1 = 0$

Or, $k = -2$

Ans : $k = -2$

b Given, $f(x) = \frac{x + 1}{3x - 1}$

Let, $f^{-1}(x) = a$

Or, $x = f(a)$

Or, $x = \frac{a + 1}{3a - 1}$

Or, $3ax - x = a + 1$

Or, $3ax - a = x + 1$

Or, $a(3x - 1) = x + 1$

Or, $a = \frac{x + 1}{3x - 1}$

$\therefore f^{-1}(x) = \frac{x + 1}{3x - 1}$

For $3x - 1 = 0$, or, $x = \frac{1}{3}$, $f^{-1}(x)$ will be undefined. But for all real number except $x = \frac{1}{3}$ the real values of $f^{-1}(x)$ will be obtained.

$\therefore$ Dom $f^{-1}(x) = \mathbb{R} - \left\{\frac{1}{3}\right\}$ **(Ans.)**

c Given,

$$g(x) = \frac{x^2 + x - 1}{x^3 - 4x} = \frac{x^2 + x - 1}{x(x^2 - 4)} = \frac{x^2 + x - 1}{x(x + 2)(x - 2)}$$

Let, $\frac{x^2 + x - 1}{x^3 - 4x} \equiv \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x - 2}$

Or, $x^2 + x - 1 = A(x + 2)(x - 2) + Bx(x - 2) + Cx(x + 2)$

For, $x = 0$,

$-1 = A.2(-2)$

Or, $A = \frac{1}{4}$

For, $x = -2$

$1 = B(-2)(-4)$

Or, $B = \frac{1}{8}$

For, $x = 2$

$5 = C.2.4$

Or, $C = \frac{5}{8}$

$\therefore \frac{x^2 + x - 1}{x^3 - 4x} = \frac{1}{4x} + \frac{1}{8(x + 2)} + \frac{5}{8(x - 2)}$ **(Ans.)**

Question ▶ 34 If $f(y) = \frac{y^3 - 2y^2 + 1}{y^2 - 2y - 3}$, then–

[Jashore English School and College (JESC), Jashore]

a. Find $f\left(-\frac{1}{3}\right)$ 2

b. If $f(y) = 0$, find the value of y. 4

c. Express f(y) as partial fractions. 4

Solution to the question no. 34

a Given, $f(y) = \frac{y^3 - 2y^2 + 1}{y^2 - 2y - 3}$

$$\therefore f\left(-\frac{1}{3}\right) = \frac{\left(-\frac{1}{3}\right)^3 - 2\left(-\frac{1}{3}\right)^2 + 1}{\left(-\frac{1}{3}\right)^2 - 2\left(-\frac{1}{3}\right) - 3} = \frac{-\frac{1}{27} - \frac{2}{9} + 1}{\frac{1}{9} + \frac{2}{3} - 3} = \frac{\frac{-1 - 6 + 27}{27}}{\frac{1 + 6 - 27}{9}} = \frac{20}{27} \times \frac{9}{(-20)} = -\frac{1}{3}$$

$\therefore f\left(-\frac{1}{3}\right) = -\frac{1}{3}$ **(Ans.)**

b If, $f(y) = 0$,

$\therefore \frac{y^3 - 2y^2 + 1}{y^2 - 2y - 3} = 0$

Or, $y^3 - 2y^2 + 1 = 0$

Let, $g(y) = y^3 - 2y^2 + 1$

$\therefore g(1) = (1)^3 - 2(1)^2 + 1 = 1 - 2 + 1 = 0$

$\therefore (y - 1)$, is a factor of g(y)

Now, $y^3 - 2y^2 + 1 = 0$

Or, $y^3 - y^2 - y^2 + y - y + 1 = 0$

Or, $y^2(y - 1) - y(y - 1) - 1(y - 1) = 0$

Or, $(y - 1)(y^2 - y - 1) = 0$

Either, $y - 1 = 0$ or, $y^2 - y - 1 = 0$

Or, $y = 1$

Comparing this equation with $ax^2 + bx + c = 0$ we get,

$a = 1, b = -1, c = -1$

$\therefore y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1(-1)}}{2.1}$

$\therefore y = \frac{1 \pm \sqrt{5}}{2}$

$\therefore$ The values of y are $1, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}$ **(Ans.)**

c The expression is $\frac{y^3 - 2y^2 + 1}{y^2 - 2y - 3}$

Now, $\frac{y^3 - 2y^2 + 1}{y^2 - 2y - 3} = \frac{y(y^2 - 2y - 3) + 3y + 1}{y^2 - 2y - 3}$

$= y + \frac{3y + 1}{(y - 3)(y + 1)}$

Now, $\frac{3y + 1}{(y - 3)(y + 1)}$ is a proper fraction.

Let, $\frac{3y + 1}{(y - 3)(y + 1)} \equiv \frac{A}{y - 3} + \frac{B}{y + 1}$ ....(i)

Multiplying $(y - 3)(y + 1)$ to both sides of (i)

$3y + 1 \equiv A(y + 1) + B(y - 3)$ ....(ii)

Putting $y = -1$ of equation (ii)

$3(-1) + 1 = A(-1 + 1) + B(-1 - 3)$

Or, $-3 + 1 = A.0 - 4B$

Or, $-2 = -4B$

$\therefore B = \frac{2}{4} = \frac{1}{2}$

Again, putting $y = 3$ of equation (ii)

$3.3 + 1 = A(3 + 1) + B(3 - 3)$

Or, $9 + 1 = 4.A + B.0$

Or, $10 = 4A$

$\therefore A = \frac{10}{4} = \frac{5}{2}$

Putting values of A and B in (i)

$$\frac{3y + 1}{(y - 3)(y + 1)} = \frac{\frac{5}{2}}{y - 3} + \frac{\frac{1}{2}}{y + 1}$$

$\therefore$ Required partial fraction,

$$\frac{y^3 - 2y^2 + 1}{y^2 - 2y - 3} = y + \frac{\frac{5}{2}}{y - 3} + \frac{\frac{1}{2}}{y + 1} = y + \frac{5}{2(y - 3)} + \frac{1}{2(y + 1)}$$ **(Ans.)**