Chapter-7: Infinite Series

Question ▶ 1 (i) $\frac{1}{2x+1}+\frac{1}{(2x+1)^2}+\frac{1}{(2x+1)^3}+$ .... is an infinite geometric series. [All Board-18]

(ii) $\left(2+\frac{x}{4}\right)^6$ and $\left(k+\frac{y}{3}\right)^7$ are two binomial expressions.

a. Expand 1st binomial expression up to x^3. 2

b. If the coefficient of k^3 is 560, then determine the value of y. 4

c. If the sum to the infinity of the infinite series given in the stem is exist, then find it. 4

Solution to the question no. 1

a 1st binomial expression, $\left(2+\frac{x}{4}\right)^6$

$$\left(2+\frac{x}{4}\right)^6=2^6+\binom{6}{1}2^5.\frac{x}{4}+\binom{6}{2}.2^4.\left(\frac{x}{4}\right)^2+\binom{6}{3}2^3\left(\frac{x}{4}\right)^3+\dots\dots$$

$= 64 + 48x + 15x^2 + \frac{5}{2}x^3 + \dots\dots\dots$ **(Ans.)**

b Using binomial theorem, we get,

$$\left(k-\frac{y}{3}\right)^7=k^7+\binom{7}{1}k^6\left(-\frac{y}{3}\right)+\binom{7}{2}k^5\left(-\frac{y}{3}\right)^2+\binom{7}{3}k^4\left(-\frac{y}{3}\right)^3+\binom{7}{4}k^3\left(-\frac{y}{3}\right)^4+\dots\dots\dots$$

Here, coefficient of k^3 of the expansion $=\binom{7}{4}\left(-\frac{y}{3}\right)^4$

$=\frac{7.6.5.4}{1.2.3.4}.\frac{y^4}{3^4}$

$=\frac{35}{81}y^4$

According to question, $\frac{35}{81}y^4=560$

Or, $y^4=560\times\frac{81}{35}$

Or, $y^4=1296$

Or, $y^4=(6)^4$

$\therefore$ $y=6$ **(Ans.)**

c 1st term of the given series, $a=\frac{1}{2x+1}$

Common ratio, $r=\frac{1}{(2x+1)^2}\div\frac{1}{(2x+1)}=\frac{1}{2x+1}$

Now, sum to infinity of the series will exist if and only if $|r|<1$

Or, $\left|\frac{1}{2x+1}\right|<1$

That is, $-1<\frac{1}{2x+1}<1$

$\therefore -1<\frac{1}{2x+1}$ Or, $\frac{1}{2x+1}<1$

Or, $-1>2x+1$ Or, $2x+1>1$

Or, $-1-1>2x+1-1$ Or, $2x+1-1>1-1$

Or, $-2>2x$ Or, $2x>0$

$\therefore x<-1$ $\therefore x>0$

$\therefore$ Sum to infinity of the series will exist if $x<-1$ or $x>0$.

$\therefore$ Sum to infinity, $S_\infty=\frac{a}{1-r}$

$=\frac{\frac{1}{2x+1}}{1-\frac{1}{2x+1}}$

$=\frac{\frac{1}{2x+1}}{\frac{2x+1-1}{2x+1}}$

$=\frac{1}{2x+1}\times\frac{2x+1}{2x}$

$=\frac{1}{2x}$ **(Ans.)**

Question ▶ 2 $\frac{1}{2x-5}+\frac{1}{(2x-5)^2}+\frac{1}{(2x-5)^3}+$ .... is a series. [Dj.B.17]

a. Find the series if x = 4. What is common ratio of the obtained series? 2

b. Find the 9th term and the sum of first 10 terms of the series when x = 5. 4

c. Find the condition which should be imposed on x, so that the given series will have a sum up to infinity and find the sum. 4

Solution to the question no. 2

a Given, the series, $\frac{1}{2x-5}+\frac{1}{(2x-5)^2}+\frac{1}{(2x-5)^3}+\dots\dots\dots$

If x = 4, the series, $\frac{1}{2\times4-5}+\frac{1}{(2\times4-5)^2}+\frac{1}{(2\times4-5)^3}+\dots\dots$

$=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\dots\dots$ **(Ans.)**

$\therefore$ Common ratio $=\frac{1}{3^2}\div\frac{1}{3}=\frac{1}{3}$ **(Ans.)**

b If x = 5, the series is,

$\frac{1}{2\times5-5}+\frac{1}{(2\times5-5)^2}+\frac{1}{(2\times5-5)^3}+\dots\dots$

$=\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\dots\dots\dots$

$\therefore$ First term, $a=\frac{1}{5}$

Common ratio, $r=\frac{1}{5^2}\div\frac{1}{5}=\frac{1}{5}$

We know, nth term of the geometric series $=ar^{n-1}$

$\therefore$ 9th term $=\frac{1}{5}.\left(\frac{1}{5}\right)^{9-1}$

$=\frac{1}{5}.\frac{1}{5^8}=\frac{1}{5^9}$ **(Ans.)**

Again, sum of n terms, $S_n = \frac{a(1-r^n)}{1-r}$, when $r < 1$

$$\therefore \text{ Sum of first 10 terms, } S_{10} = \frac{\frac{1}{5}\left(1-\frac{1}{5^{10}}\right)}{1-\frac{1}{5}}$$

$$= \frac{\frac{1}{5}\cdot\frac{5^{10}-1}{5^{10}}}{\frac{4}{5}}$$

$$= \frac{1}{5}\times\frac{5}{4}\times\frac{5^{10}-1}{5^{10}}$$

$$= \frac{5^{10}-1}{4\times 5^{10}} \textbf{ (Ans.)}$$

c Given, 1st term of the series, $a = \frac{1}{2x-5}$

Common ratio, $r = \frac{1}{(2x-5)^2} \div \frac{1}{(2x-5)}$

$$= \frac{1}{2x-5}$$

Sum to infinity of the series will exist if $|r| < 1$

That is if, $\left|\frac{1}{2x-5}\right| < 1$.

$\therefore \frac{1}{2x-5} < 1$	Again, $\frac{1}{2x-5} > -1$
Or, $2x-5 > 1$	Or, $2x-5 < -1$
Or, $2x-5+5 > 1+5$	Or, $2x-5+5 < -1+5$
Or, $2x > 6$	Or, $2x < 4$
$\therefore x > 3$	$\therefore x < 2$

$\therefore$ Required condition, $x > 3$ or, $x < 2$ **(Ans.)**

Now, sum to infinity of the series, $S_\infty = \frac{a}{1-r}$

$$= \frac{\frac{1}{2x-5}}{1-\frac{1}{2x-5}}$$

$$= \frac{1}{2x-5}\times\frac{2x-5}{2x-6}$$

$$= \frac{1}{2x-6} \textbf{ (Ans.)}$$

Question ▸ 3 Consider the following infinite series :

$1 + \frac{1}{1+3x} + \frac{1}{(1+3x)^2} + \frac{1}{(1+3x)^3} + \ldots\ldots\ldots\ldots$ [C.B.17]

a. Find the common ratio of the obtained series when x = 1. 2

b. Find the sum of first 10 terms of the series obtained in $x = \frac{1}{3}$. 4

c. Find the condition which should be imposed on x, so that the given series with have a sum up to infinity. 4

<u>Solution to the question no. 3</u>

a Given, the series, $1 + \frac{1}{1+3x} + \frac{1}{(1+3x)^2} + \frac{1}{(1+3x)^3} + \ldots\ldots$

If x = 1, the series,

$$1 + \frac{1}{1+3.1} + \frac{1}{(1+3.1)^2} + \frac{1}{(1+3.1)^3} + \ldots\ldots$$

$$= 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\ldots$$

$$\therefore \text{ Common ratio} = \frac{2^{nd}\text{ term}}{1^{st}\text{ term}} = \frac{\frac{1}{4}}{1} = \frac{1}{4} \textbf{ (Ans.)}$$

b If $x = \frac{1}{3}$, the series,

$$1 + \frac{1}{1+3.\frac{1}{3}} + \frac{1}{\left(1+3.\frac{1}{3}\right)^2} + \frac{1}{\left(1+3.\frac{1}{3}\right)^3} + \ldots\ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$$

Whose 1st term, a = 1

Common ratio, $r = \frac{\frac{1}{2}}{1} = \frac{1}{2} < 1$

We know, sum of 1st n terms of a geometric series,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\therefore \text{ Sum of 1st 1o terms of the series, } S_{10} = \frac{a(1-r^{10})}{1-r}$$

$$= \frac{1\left\{1-\left(\frac{1}{2}\right)^{10}\right\}}{1-\frac{1}{2}}$$

$$= \frac{\frac{2^{10}-1}{2^{10}}}{\frac{1}{2}}$$

$$= \frac{1024-1}{1024}\times 2$$

$$= \frac{1023}{512} \textbf{ (Ans.)}$$

c Here, common ratio of the series, $r = \frac{\frac{1}{1+3x}}{1} = \frac{1}{1+3x}$

Sum to infinity of the series will exist is $|r| < 1$

Or, $\left|\frac{1}{1+3x}\right| < 1$

That is, $-1 < \frac{1}{1+3x} < 1$

$\therefore -1 < \frac{1}{1+3x}$	Or, $\frac{1}{1+3x} < 1$
Or, $-1 > 1+3x$	Or, $1+3x > 1$
Or, $-1-1 > 1+3x-1$	Or, $1+3x-1 > 1-1$
Or, $-2 > 3x$	Or, $3x > 0$
$\therefore x < -\frac{2}{3}$	$\therefore x > 0$

$\therefore$ Required conditions: $x > 0$ or $x < -\frac{2}{3}$ **(Ans.)**

Question ▸ 4 $\frac{1}{3x-1} + \frac{1}{(3x-1)^2} + \frac{1}{(3x-1)^3} + \ldots\ldots\ldots$ [S.B.17]

a. If x = 2, then determine the common ratio of the series. 2

b. Determine the 7th term of the series and summation of first ten terms of the series when x=1. 4

c. Impose the condition on x so that the series will have an infinite sum and find that sum. 4

Solution to the question no. 4

a Given, the series, $\frac{1}{3x-1}+\frac{1}{(3x-1)^2}+\frac{1}{(3x-1)^3}+.........$

If $x=2$, the series, $\frac{1}{3.2-1}+\frac{1}{(3.2-1)^2}+\frac{1}{(3.2-1)^3}+....$

$=\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+..................$

$\therefore$ Common ratio of the series $=\frac{1}{5^2}\div\frac{1}{5}$

$=\frac{1}{5}$ **(Ans.)**

b If $x=1$, the series, $\frac{1}{3.1-1}+\frac{1}{(3.1-1)^2}+\frac{1}{(3.1-1)^3}+.....$

$=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+..................$

Whose 1^{st} term, $a=\frac{1}{2}$

Common ratio, $r=\frac{1}{2^2}\div\frac{1}{2}=\frac{1}{2}<1$

We know, n^{th} term of the geometric series $=ar^{n-1}$

$\therefore$ 7^{th} term $=\frac{1}{2}\left(\frac{1}{2}\right)^{7-1}=\frac{1}{2}\cdot\frac{1}{2^6}=\frac{1}{2^7}$ **(Ans.)**

Again, sum of 1^{st} n terms of the geometric series,

$$S_n=\frac{a(1-r^n)}{1-r}$$

$\therefore$ Sum of 1^{st} 10 terms, $S_{10}=\frac{\frac{1}{2}\left(1-\frac{1}{2^{10}}\right)}{1-\frac{1}{2}}$

$=\frac{\frac{1}{2}\left(\frac{2^{10}-1}{2^{10}}\right)}{\frac{1}{2}}$

$=\frac{1024-1}{1024}=\frac{1023}{1024}$ **(Ans.)**

c Given the series, $\frac{1}{3x-1}+\frac{1}{(3x-1)^2}+\frac{1}{(3x-1)^3}+.........$

1st term of the series, $a=\frac{1}{3x-1}$

Common ratio, $r=\frac{1}{(3x-1)^2}\div\frac{1}{3x-1}$

$=\frac{1}{(3x-1)^2}\times\frac{3x-1}{1}$

$=\frac{1}{3x-1}$

Sum to infinity of the series will exist if $|r|<1$.

That is, $\left|\frac{1}{3x-1}\right|<1$

Or, $-1<\frac{1}{3x-1}<1$

$\therefore \frac{1}{3x-1}<1$

Or, $3x-1>1$

Or, $3x-1+1>1+1$

Or, $3x>2$

$\therefore x>\frac{2}{3}$

Or, $\frac{1}{3x-1}>-1$

Or, $3x-1<-1$

Or, $3x<-1+1$

Or, $3x<0$

$\therefore x<0$

$\therefore$ Sum to infinity of the series will exist if $x>\frac{2}{3}$ or $x<0$. **(Ans.)**

Now, sum to infinity of the series,

$$S_\infty=\frac{a}{1-r}=\frac{\frac{1}{3x-1}}{1-\frac{1}{3x-1}}=\frac{\frac{1}{3x-1}}{\frac{3x-1-1}{3x-1}}=\frac{1}{3x-2}\text{ (Ans.)}$$

Question ▶ 5 $\frac{1}{2x+1}+\frac{1}{(2x+1)^2}+\frac{1}{(2x+1)^3}+.........$ [J.B.17]

a. If $x=3$, find the common ratio of the series. 2

b. If $x=-2$, find the 10th term of the series and sum of first 10 terms. 4

c. Impose a condition on x under which the series will have a sum up to infinity and find the sum. 4

Solution to the question no. 5

a Given series $\frac{1}{2x+1}+\frac{1}{(2x+1)^2}+\frac{1}{(2x+1)^3}+................$

If $x=3$, the series, $\frac{1}{2.3+1}+\frac{1}{(2.3+1)^2}+\frac{1}{(2.3+1)^3}+..........$

$=\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+............$

$\therefore$ Common ratio, $r=\frac{1}{7^2}\div\frac{1}{7}$

$=\frac{1}{7}$ **(Ans.)**

b If $x=-2$, the series,

$\frac{1}{2.(-2)+1}+\frac{1}{\{2.(-2)+1\}^2}+\frac{1}{\{2.(-2)+1\}^3}+...........$

$=-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+........$

1^{st} term of the series, $a=-\frac{1}{3}$

Common ratio, $r=\frac{1}{3^2}\div\frac{-1}{3}=-\frac{1}{3}$

We know, n^{th} term of the geometric series $=ar^{n-1}$

$\therefore$ 10^{th} term of the series $=-\frac{1}{3}\cdot\left(-\frac{1}{3}\right)^{10-1}$

$=\frac{1}{3}\cdot\frac{1}{3^9}$

$=\frac{1}{3^{10}}$ **(Ans.)**

Again, sum of 1^{st} n terms of the geometric series,

$$S_n=\frac{a(1-r^n)}{1-r}$$

$\therefore$ Sum of 1^{st} 8 terms, $S_8=\frac{\frac{-1}{3}\left\{1-\left(-\frac{1}{3}\right)^8\right\}}{1-\left(-\frac{1}{3}\right)}$

$=\frac{\frac{-1}{3}\left(1-\frac{1}{3^8}\right)}{1+\frac{1}{3}}$

$=\frac{-1}{3}\times\frac{3}{4}\times\frac{3^8-1}{3^8}$

$=\frac{1-3^8}{4\times3^8}$ **(Ans.)**

c Given, 1st term of the series, $a = \frac{1}{2x+1}$

Common ratio, $r = \frac{1}{(2x+1)^2} \div \frac{1}{(2x+1)}$

$= \frac{1}{2x+1}$

Now, sum to infinity of the series will exist if and only if

$|r| < 1$ or, $\left|\frac{1}{2x+1}\right| < 1.$

That is $-1 < \frac{1}{2x+1} < 1$

$\therefore -1 < \frac{1}{2x+1}$ | Or, $\frac{1}{2x+1} < 1$

Or, $-1 > 2x + 1$ | Or, $2x + 1 > 1$

Or, $-1 - 1 > 2x + 1 - 1$ | Or, $2x + 1 - 1 > 1 - 1$

Or, $-2 > 2x$ | Or, $2x > 0$

$\therefore x < -1$ | $\therefore x > 0$

$\therefore$ The required conditions: $x < -1$ or, $x > 0$ **(Ans.)**

Again, sum to infinity of the series, $S_\infty = \frac{a}{1-r}$

$$= \frac{\frac{1}{2x+1}}{1 - \frac{1}{2x+1}}$$

$$= \frac{\frac{1}{2x+1}}{\frac{2x+1-1}{2x+1}}$$

$$= \frac{1}{2x+1} \times \frac{2x+1}{2x}$$

$$= \frac{1}{2x} \text{ (Ans.)}$$

Question ▶6 Consider the following series :

$\frac{1}{3x-1} + \frac{1}{(3x-1)^2} + \frac{1}{(3x-1)^3} + \dots \dots$

[Sylhet Cadet College, Sylhet]

a. Find the series if x = 2, what is common ratio of the obtained series? 2

b. Find the 15th term and the sum of first 15 terms of the series obtained in (a). 4

c. Find the condition which should be imposed on x, so that given series will have a sum up to infinity and find the sum. 4

Solution to the question no. 6

a Given, the series, $\frac{1}{3x-1} + \frac{1}{(3x-1)^2} + \frac{1}{(3x-1)^3} + \dots\dots\dots$

If x = 2, the series, $\frac{1}{3.2-1} + \frac{1}{(3.2-1)^2} + \frac{1}{(3.2-1)^3} + \dots$

$= \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots\dots\dots\dots$

$\therefore$ Common ratio of the series $= \frac{1}{5^2} \div \frac{1}{5} = \frac{1}{5}$ **(Ans.)**

b From 'a', the series is

$\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots\dots\dots$

1st term, $a = \frac{1}{5}$

Common ratio, $r = \frac{1}{5} < 1$

We know that, nth term of a geometric series $= ar^{n-1}$

$\therefore$ 15th term of the series $= ar^{15-1} = \frac{1}{5} \cdot \left(\frac{1}{5}\right)^{14} = \frac{1}{5^{15}}$ **(Ans.)**

Also we know that, sum of first nth term of a geometric series $= \frac{a(1-r^n)}{1-r}$, $r < 1$.

$\therefore$ Sum of 1st 15 term,

$$S_{15} = \frac{a(1-r^{15})}{1-r} = \frac{1}{5}\left\{\frac{1-\left(\frac{1}{5}\right)^{15}}{1-\frac{1}{5}}\right\}$$

$$= \frac{\frac{1}{5}\left\{1 - \frac{1}{5^{15}}\right\}}{\frac{4}{5}}$$

$$= \frac{1}{4}\left(\frac{5^{15}-1}{5^{15}}\right) \text{ (Ans.)}$$

c Given, the series, $\frac{1}{3x-1} + \frac{1}{(3x-1)^2} + \frac{1}{(3x-1)^3} + \dots\dots\dots$

1st term of the series, $a = \frac{1}{3x-1}$

Common ratio, $r = \frac{1}{(3x-1)^2} \div \frac{1}{3x-1}$

$= \frac{1}{(3x-1)^2} \times \frac{3x-1}{1}$

$= \frac{1}{3x-1}$

Sum to infinity of the series will exist if $|r| < 1$.

That is, $\left|\frac{1}{3x-1}\right| < 1$

Or, $-1 < \frac{1}{3x-1} < 1$

$\therefore \frac{1}{3x-1} < 1$ | Or, $\frac{1}{3x-1} > -1$

Or, $3x - 1 > 1$ | Or, $3x - 1 < -1$

Or, $3x - 1 + 1 > 1 + 1$ | Or, $3x - 1 + 1 < -1 + 1$

Or, $3x > 2$ | Or, $3x < 0$

$\therefore x > \frac{2}{3}$ | $\therefore x < 0$

$\therefore$ Sum to infinity of the series will exist if $x > \frac{2}{3}$ or $x < 0$. **(Ans.)**

Now, sum to infinity of the series,

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{3x-1}}{1-\frac{1}{3x-1}} = \frac{\frac{1}{3x-1}}{\frac{3x-1-1}{3x-1}} = \frac{1}{3x-2} \text{ (Ans.)}$$

Question ▶7 $1 + (1+3x)^{-1} + (1+3x)^{-2} + (1+3x)^{-3} \dots \dots \dots$ is an infinite geometric series and $\left(2+\frac{x}{4}\right)^6$ is binomial expression.

[Milestone College, Dhaka]

a. If x = 1, then find the series and find its common ratio. 2

b. Expand the binomial upto x^3 in ascending power of x and by using the result find the value of $(1.9975)^6$ upto four decimal places. 4

c. Imposing which condition on x the given series will have a sum upto infinity and find the sum. 4

Solution to the question no. 7

a Given, the series, $1+\frac{1}{1+3x}+\frac{1}{(1+3x)^2}+\frac{1}{(1+3x)^3}+ \dots\dots$

If x = 1, the series,

$$1+\frac{1}{1+3.1}+\frac{1}{(1+3.1)^2}+\frac{1}{(1+3.1)^3}+ \dots\dots$$

$$= 1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+ \dots\dots$$

$\therefore$ Common ratio, $r = \frac{2^{nd} \text{ term}}{1^{st} \text{ term}}$

$$= \frac{\frac{1}{4}}{1}$$

$$= \frac{1}{4} \textbf{ (Ans.)}$$

b By using binomial theorem, we get,

$$\left(2+\frac{x}{4}\right)^6 = 2^6 + \binom{6}{1}.2^5.\left(\frac{x}{4}\right) + \binom{6}{2}.2^4.\left(\frac{x}{4}\right)^2 + \binom{6}{3}.2^3.\left(\frac{x}{4}\right)^3 + \dots$$

$$= 64 + 6.32.\frac{x}{4} + \frac{6.5}{1.2}.16.\frac{x^2}{16} + \frac{6.5.4}{1.2.3}.8.\frac{x^3}{64} + \dots$$

$$= 64 + 48x + 15x^2 + \frac{5}{2}x^3 + \dots \textbf{ (Ans.)}$$

Here, $2+\frac{x}{4} = 1.9975$

Or, $\frac{x}{4} = 1.9975 - 2 = -0.0025$

Or, $x = (-0.0025) \times 4$

$\therefore x = -0.01$

Here, putting $x = -0.01$ we get,

$$\left\{2+\frac{(-0.01)}{4}\right\}^6 = 64 + 48(-0.01) + 15.(-0.01)^2 + \frac{5}{2}(-0.01)^3 + \dots\dots$$

$\therefore (1.9975)^6 = 63.5215$; [upto four decimal places] **(Ans.)**

c Here, common ratio of the series, $r = \frac{\frac{1}{1+3x}}{1} = \frac{1}{1+3x}$

Sum to infinity of the series will exist is $|r| < 1$

Or, $\left|\frac{1}{1+3x}\right| < 1$

That is, $-1 < \frac{1}{1+3x} < 1$

$\therefore -1 < \frac{1}{1+3x}$	Or, $\frac{1}{1+3x} < 1$
Or, $-1 > 1+3x$	Or, $1+3x > 1$
Or, $-1-1 > 1+3x-1$	Or, $1+3x-1 > 1-1$
Or, $-2 > 3x$	Or, $3x > 0$
$\therefore x < -\frac{2}{3}$	$\therefore x > 0$

$\therefore$ Required conditions: $x > 0$ or, $x < -\frac{2}{3}$ **(Ans.)**

Now, sum to infinity of the series,

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-\frac{1}{1+3x}}$$

$$= \frac{1}{\frac{1+3x-1}{1+3x}}$$

$$= \frac{1+3x}{3x}$$

$$= 1+\frac{1}{3x} \textbf{ (Ans.)}$$

Question ▸ 8 $\frac{1}{2x+3}+\frac{1}{(2x+3)^2}+\frac{1}{(2x+3)^3}+ \dots\dots\dots$ is an infinite series. *[Saint Joseph Higher Secondary School, Dhaka]*

a. Find the series and the common ratio of the series when x = 2. 2

b. Determine the 8^{th} term of the series and summation up to 10^{th} terms of the series when $x = \frac{1}{2}$. 4

c. Impose the condition x so that the series will have an infinite sum and find the sum. 4

Solution to the question no. 8

a Given, series, $\frac{1}{2x+3}+\frac{1}{(2x+3)^2}+\frac{1}{(2x+3)^3}+ \dots\dots$

If x = 2, the series, $\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+ \dots\dots$ **(Ans.)**

And common ratio, $r = \frac{1}{7^2} \div \frac{1}{7}$

$$= \frac{1}{7} \textbf{ (Ans.)}$$

b When $x = \frac{1}{2}$,

then the series, $\frac{1}{2.\frac{1}{2}+3}+\frac{1}{(2.\frac{1}{2}+3)^2}+\frac{1}{(2.\frac{1}{2}+3)^3}+ \dots\dots$

$$= \frac{1}{1+3}+\frac{1}{(1+3)^2}+\frac{1}{(1+3)^3}+ \dots\dots$$

$$= \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+ \dots\dots$$

Here, 1^{st} term, $a = \frac{1}{4}$

Common ratio, $r = \frac{1}{4^2} \div \frac{1}{4}$

$$= \frac{1}{4} < 1$$

$\therefore 8^{th}$ term $= ar^{8-1}$

$$= \frac{1}{4}.\left(\frac{1}{4}\right)^7$$

$$= \left(\frac{1}{4}\right)^8$$

$$= \frac{1}{65536} \textbf{ (Ans.)}$$

Sum of first 10^{th} term $= a.\frac{1-r^n}{1-r}$ $[\because r < 1]$

$$= \frac{1}{4} \times \frac{1-\left(\frac{1}{4}\right)^{10}}{1-\frac{1}{4}}$$

$$= \frac{1}{4} \times \frac{1 - \frac{1}{4^{10}}}{\frac{3}{4}}$$

$$= \frac{4^{10} - 1}{3 \times 4^{10}}$$

$$= \frac{1048575}{3 \times 1048576}$$

$$= \frac{1048575}{3145728} \text{ (Ans.)}$$

c The series, $\frac{1}{2x+3} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+3)^3} + ...$

Common ratio, $r = \frac{1}{(2x+3)^2} \div \frac{1}{2x+3}$

$= \frac{1}{2x+3}$

Now, the sum upto infinity of the series will be possible if

$-1 < \frac{1}{2x+3} < 1$

That is, $-1 < \frac{1}{2x+3}$

Or, $2x + 3 < -1$

Or, $2x < -4$

Or, $x < -2$

Again, $\frac{1}{2x+3} < 1$

Or, $1 < 2x + 3$

Or, $2x > 1 - 3$

Or, $x > \frac{-2}{2}$

Or, $x > -1$

$\therefore$ The conditions: $x < -2$, Or $x > -1$

and the sum of the series to the infinity,

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{1}{2x+3}}{1 - \frac{1}{2x+3}}$$

$$= \frac{\frac{1}{2x+3}}{\frac{2x+3-1}{2x+3}}$$

$$= \frac{1}{2x+3} \times \frac{2x+3}{2x+2}$$

$$= \frac{1}{2x+2} \text{ (Ans.)}$$

Question ▸ 9 $\frac{1}{2x+1} + \frac{1}{(2x+1)^2} + \frac{1}{(2x+1)^3} + ...$

[Mirpur Girls' Ideal Laboratory Institute, Dhaka]

a. If $x = 2$, then find the common ratio of the series. 2

b. If $x = -2$ find the 7^{th} term of the series and sum of first 10 terms. 4

c. Impose a condition on x under which the series will have a sum up to infinity and find the sum. 4

Solution to the question no. 9

a Given, the series,

$\frac{1}{2x+1} + \frac{1}{(2x+1)^2} + \frac{1}{(2x+1)^3} + ...$

If $x = 2$, the series $\frac{1}{2.2+1} + \frac{1}{(2.2+1)^2} + \frac{1}{(2.2+1)^3}$

$= \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ...$

$\therefore$ Common ratio $= \frac{1}{5^2} \div \frac{1}{5} = \frac{1}{5}$ **(Ans.)**

b If $x = -2$, the series,

$\frac{1}{2.(-2)+1} + \frac{1}{\{2.(-2)+1\}^2} + \frac{1}{\{2.(-2)+1\}^3} + ...$

$= -\frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + ...$

1^{st} term of the series, $a = -\frac{1}{3}$

Common ratio, $r = \frac{1}{3^2} \div \frac{-1}{3} = -\frac{1}{3}$

We know, n^{th} term of the geometric series $= ar^{n-1}$

$\therefore 7^{th}$ term of the series $= -\frac{1}{3} . \left(-\frac{1}{3}\right)^{7-1}$

$= -\frac{1}{3} . \frac{1}{3^6}$

$= -\frac{1}{3^7}$ **(Ans.)**

Again, sum of 1^{st} n terms of the geometric series,

$S_n = \frac{a(1-r^n)}{1-r}$

$$\therefore \text{Sum of } 1^{st} \text{ 10 terms, } S_{10} = \frac{\frac{-1}{3}\left\{1 - \left(-\frac{1}{3}\right)^{10}\right\}}{1 - \left(-\frac{1}{3}\right)}$$

$$= \frac{\frac{-1}{3}\left(1 - \frac{1}{3^{10}}\right)}{1 + \frac{1}{3}}$$

$$= \frac{-1}{3} \times \frac{3}{4} \times \frac{3^{10} - 1}{3^{10}}$$

$$= \frac{1 - 3^{10}}{4 \times 3^{10}} \text{ (Ans.)}$$

c Given, 1^{st} term of the series, $a = \frac{1}{2x+1}$

Common ratio, $r = \frac{1}{(2x+1)^2} \div \frac{1}{(2x+1)}$

$= \frac{1}{2x+1}$

Now, sum to infinity of the series will exist if and only if

$|r| < 1$ or, $\left|\frac{1}{2x+1}\right| < 1$.

That is $-1 < \frac{1}{2x+1} < 1$

$\therefore -1 < \frac{1}{2x+1}$

Or, $-1 > 2x + 1$

Or, $-1 - 1 > 2x + 1 - 1$

Or, $\frac{1}{2x+1} < 1$

Or, $2x + 1 > 1$

Or, $2x + 1 - 1 > 1 - 1$

Or, $-2 > 2x$ Or, $2x > 0$

$\therefore x < -1$ $\therefore x > 0$

$\therefore$ The required conditions: $x < -1$ or, $x > 0$ **(Ans.)**

Again, sum to infinity of the series, $S_\infty = \dfrac{a}{1-r}$

$$= \frac{\frac{1}{2x+1}}{1 - \frac{1}{2x+1}}$$

$$= \frac{\frac{1}{2x+1}}{\frac{2x+1-1}{2x+1}}$$

$$= \frac{1}{2x+1} \times \frac{2x+1}{2x}$$

$$= \frac{1}{2x} \textbf{ (Ans.)}$$

Question ▶ 10 Consider the following series: $\dfrac{1}{8x+1} + \dfrac{1}{(8x+1)^2} + \dfrac{1}{(8x+1)^3} + \ldots\ldots\ldots\ldots$ *[Chetona Model Academy (CMA), Dhaka]*

a. Find the sum of the series $2 + 4 + 8 + 16 + \ldots\ldots\ldots\ldots$ 2

b. Find the series if $x = 1$, also find the sum of the first 10 term of the series. 4

c. Impose a condition on 'x' under which the given infinity series will have a sum and find that sum. 4

Solution to the question no. 10

a $2 + 4 + 8 + 16 + \ldots\ldots$

Here, the series is a infinite series and first term, $a = 2$

Common ratio, $r = 2 > 1$

Hence, the sum of the series does not exist.

b If $x = 1$, then the given series,

$$\frac{1}{8.1+1} + \frac{1}{(8.1+1)^2} + \frac{1}{(8.1+1)^3} + \ldots\ldots$$

$$= \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \ldots\ldots \textbf{ (Ans.)}$$

Here, first term, $a = \dfrac{1}{9} < 1$

Common ratio, $r = \dfrac{1}{9}$

$\therefore$ Sum of 10 term, $S_{10} = \dfrac{\frac{1}{9}\left\{1 - \left(\frac{1}{9}\right)^{10}\right\}}{1 - \frac{1}{9}}$

$$= \frac{\frac{1}{9}\left\{\frac{9^{10}-1}{9^{10}}\right\}}{\frac{9-1}{9}}$$

$$= \frac{9^{10}-1}{8 \times 9^{10}} \textbf{ (Ans.)}$$

c First term of the given series, $a = \dfrac{1}{8x+1}$

And common ratio, $r = \dfrac{1}{(8x+1)^2} \div \dfrac{1}{8x+1} = \dfrac{1}{8x+1}$

Sum of the given infinite series will exist if $|r| < 1$

That is, $\left|\dfrac{1}{8x+1}\right| < 1$ or, $-1 < \dfrac{1}{8x+1} < 1$

Now, $-1 < \dfrac{1}{8x+1}$

Or, $\dfrac{1}{-1} > 8x + 1$; [by making inverse]

Or, $-1 - 1 > 8x + 1 - 1$; [adding (-1) on both sides]

Or, $-2 > 8x$

Or, $-\dfrac{1}{4} > x$; [multiplying both sides by $\dfrac{1}{8}$]

$\therefore x < -\dfrac{1}{4}$

Again, $\dfrac{1}{8x+1} < 1$

Or, $8x + 1 > 1$

Or, $8x > 1 - 1$; [adding (-1) on both sides]

Or, $8x > 0$

$\therefore x > 0$

$\therefore$ The required condition : $x < -\dfrac{1}{4}$ or $x > 0$. **(Ans.)**

$$S_\infty = \frac{a}{1-r} + \frac{\frac{1}{8x+1}}{1 - \frac{1}{8x+1}}$$

$$= \frac{\frac{1}{8x+1}}{\frac{8x}{8x+1}}$$

$$= \frac{1}{8x+1} \times \frac{8x+1}{8x}$$

$$= \frac{1}{8x} \textbf{ (Ans.)}$$

Question ▶ 11 $\dfrac{1}{2x-5} + \dfrac{1}{(2x-5)^2} + \dfrac{1}{(2x-5)^3} + \ldots\ldots$ is a series.

[Dinajpur Laboratory School & College, Dinajpur]

a. Find the series of $x = 4$. What is common ratio of the obtained series? 2

b. Find the 9^{th} term and the sum of first 10 terms of the series when $x = 5$. 4

c. Find the condition which should be imposed on x, show that the given series will have a sum up to infinity and find the sum. 4

Solution to the question no. 11

a Given, the series, $\dfrac{1}{2x-5} + \dfrac{1}{(2x-5)^2} + \dfrac{1}{(2x-5)^3} + \ldots\ldots\ldots$

If $x = 4$, the series,

$$\frac{1}{2\times4-5} + \frac{1}{(2\times4-5)^2} + \frac{1}{(2\times4-5)^3} + \ldots\ldots$$

$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots\ldots \textbf{ (Ans.)}$$

$\therefore$ Common ratio $= \dfrac{1}{3^2} \div \dfrac{1}{3} = \dfrac{1}{3}$ **(Ans.)**

b If $x = 5$, the series, $\dfrac{1}{2\times5-5} + \dfrac{1}{(2\times5-5)^2} + \dfrac{1}{(2\times5-5)^3} + \ldots\ldots$

$$= \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots\ldots\ldots$$

$\therefore$ First term, $a = \frac{1}{5}$

Common ratio, $r = \frac{1}{5^2} \div \frac{1}{5} = \frac{1}{5}$

We know, n^{th} term of the geometric series $= ar^{n-1}$

$\therefore 9^{th}$ term $= \frac{1}{5} \cdot \left(\frac{1}{5}\right)^{9-1}$

$= \frac{1}{5} \cdot \frac{1}{5^8} = \frac{1}{5^9}$ **(Ans.)**

Again, sum of n^{th} terms, $S_n = \frac{a(1-r^n)}{1-r}$, when $r < 1$

$\therefore$ Sum of first 10^{th} terms, $S_{10} = \frac{\frac{1}{5}\left(1 - \frac{1}{5^{10}}\right)}{1 - \frac{1}{5}}$

$= \frac{\frac{1}{5} \cdot \frac{5^{10}-1}{5^{10}}}{\frac{4}{5}}$

$= \frac{1}{5} \times \frac{5}{4} \times \frac{5^{10}-1}{5^{10}}$

$= \frac{5^{10}-1}{4 \times 5^{10}}$ **(Ans.)**

c Given, 1^{st} term of the series, $a = \frac{1}{2x-5}$

Common ratio, $r = \frac{1}{(2x-5)^2} \div \frac{1}{(2x-5)}$

$= \frac{1}{2x-5}$

Sum to infinity of the series will exist if $|r| < 1$

That is if, $\left|\frac{1}{2x-5}\right| < 1$.

$\therefore \frac{1}{2x-5} < 1$

Or, $2x - 5 > 1$

Or, $2x - 5 + 5 > 1 + 5$

Or, $2x > 6$

$\therefore x > 3$

Again, $\frac{1}{2x-5} > -1$

Or, $2x - 5 < -1$

Or, $2x - 5 + 5 < -1 + 5$

Or, $2x < 4$

$\therefore x < 2$

$\therefore$ Required condition : $x > 3$ Or, $x < 2$ **(Ans.)**

Now, sum to infinity of the series, $S_\infty = \frac{a}{1-r}$

$= \frac{\frac{1}{2x-5}}{1 - \frac{1}{2x-5}}$

$= \frac{1}{2x-5} \times \frac{2x-5}{2x-6}$

$= \frac{1}{2x-6}$ **(Ans.)**

Question ▶ 12 $(1 + ax)^6$ is a binomial expression and $S = 4 + 44 + 444 + ...$ is a series. *[Cantonment Public School & College, Saidpur]*

a. Find the domain of $\sqrt{x-3}$. 2

b. If the coefficient of x^2 and x^4 are equal of the binomial expression, then find the value of a. 4

c. Find the sum of the series upto n terms. 4

Solution to the question no. 12

a Let, $f(x) = \sqrt{x-3}$

The function f(x) will be defined if and only if $x - 3 \geq 0$

Or, $x \geq 3$

$\therefore$ Domain $= \{x \in \mathbb{R} : x \geq 3\}$ **(Ans.)**

b Let, $g(x) = (1 + ax)^6$

Expanding g(x) binomially we get,

$(1 + ax)^6 = 1 + 6_{C_1}\, ax + 6_{C_2}\, (ax)^2 + 6_{C_3}\, (ax)^3 + 6_{C_4}\, (ax)^4 + 6_{C_5}\, (ax)^5 + (ax)^6$

$= 1 + 6ax + 15a^2x^2 + 20a^3x^3 + 15a^4x^4 + 6a^5x^5 + a^6x^6$

The coefficient of $x^2 = 15a^2$

and then " " $x^4 = 15a^4$

According to the question,

$15a^2 = 15a^4$

Or, $a^2 = a^4$

Or, $a^2 = 1$

$\therefore a = \pm 1$ **(Ans.)**

c Given series,

$S = 4 + 44 + 444 + ...$

$= 4 + 44 + 444 + ... + n^{th}$ term

$= 4(1 + 11 + 111 + ... + n^{th}$ term$)$

$= \frac{4}{9}(9 + 99 + 999 + ... + n^{th}$ term$)$

$= \frac{4}{9}\{(10-1) + (100-1) + (1000-1) + ... + n^{th}$ term$\}$

$= \frac{4}{9}\{(10 + 100 + 1000 + ... + nth$ term$) - (1+1+1+...+n^{th}$ term$)\}$

$= \frac{4}{9}\{(10 + 10^2 + 10^3 + ... + n^{th}$ term$) - n\}$

$= \frac{4}{9}\left(10 . \frac{10^n - 1}{10 - 1} - n\right)$

$= \frac{40}{81}(10^n - 1) - \frac{4n}{9}$ **(Ans.)**

Question ▶ 13 $\frac{1}{4x-5} + \frac{1}{(4x-5)^2} + \frac{1}{(4x-5)^3} + ...\ ....\ ....$ is an infinite series. *[Cantonment English School & College, Chattogram]*

a. Find the series if x = 2. What is the common ratio of the obtained series? 2

b. Find the 10^{th} term and the sum of first 10 terms of the series when x = 3. 4

c. Impose a condition on 'x' under which the given infinite series will have a sum and find that sum. 4

Solution to the question no. 13

a Given, series, $\frac{1}{4x-5} + \frac{1}{(4x-5)^2} + \frac{1}{(4x-5)^3} + ...$

If x = 2, the series

$\frac{1}{4.2-5} + \frac{1}{(4.2-5)^2} + \frac{1}{(4.2-5)^3} + ...$

$= \frac{1}{8-5} + \frac{1}{(8-5)^2} + \frac{1}{(8-5)^3} + ...$

$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...$

Common ratio, $r = \frac{1}{3^2} \div \frac{1}{3} = \frac{1}{3}$ **(Ans.)**

b If x = 3 the series

$$\frac{1}{4.3-5}+\frac{1}{(4.3-5)^2}+\frac{1}{(4.3-5)^3}+\ldots\ldots\ldots$$

$$=\frac{1}{12-5}+\frac{1}{(12-5)^2}+\frac{1}{(12-5)^3}+\ldots\ldots\ldots$$

$$=\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\ldots\ldots\ldots$$

Whose 1st term, $a=\frac{1}{7}$

Common ratio, $r=\frac{1}{7^2}\div\frac{1}{7}=\frac{1}{7}$

We know, n^{th} term of a geometric series $= ar^{n-1}$

$\therefore 10^{th}$ term $=\frac{1}{7}\left(\frac{1}{7}\right)^{10-1}=\frac{1}{7}\cdot\frac{1}{7^9}$

$=\frac{1}{7^{10}}$ **(Ans.)**

Also we know that, sum of first n^{th} term of the geometric series, $S_n=\frac{a(1-r^n)}{1-r}$

$$=\frac{\frac{1}{7}\left(1-\frac{1}{7^{10}}\right)}{1-\frac{1}{7}}$$

$$=\frac{\frac{1}{7}\left(\frac{7^{10}-1}{7^{10}}\right)}{\frac{6}{7}}$$

$$=\frac{1}{7}\times\frac{7^{10}-1}{7^{10}}\times\frac{7}{6}$$

$=\frac{7^{10}-1}{6.7^{10}}$ **(Ans.)**

c Given series, $\frac{1}{4x-5}+\frac{1}{(4x-5)^2}+\frac{1}{(4x-5)^3}+\ldots\ldots$

1st term of the series, $a=\frac{1}{4x-5}$

Common ratio, $r=\frac{1}{(4x-5)^2}\div\frac{1}{4x-5}$

$=\frac{1}{4x-5}$

Sum to the infinite series will be exist, if $|r|<1$

That is, $\left|\frac{1}{4x-5}\right|<1$

Or, $-1<\frac{1}{4x-5}<1$

$\therefore \frac{1}{4x-5}<1$ Or, $\frac{1}{4x-5}>-1$

Or, $4x-5>1$ Or, $4x-5<-1$

Or, $4x-5+5>1+5$ Or, $4x-5+5<-1+5$

Or, $4x>6$ Or, $4x<4$

Or, $\frac{4x}{4}>\frac{6}{4}$ Or, $\frac{4x}{4}<\frac{4}{4}$

$\therefore x>\frac{3}{2}$ $\therefore x<1$

$\therefore$ Sum to infinity of the series will exist if $x>\frac{3}{2}$ or, $x<1$ **(Ans.)**

Now, sum to infinity of the series,

$$S_\infty=\frac{a}{1-r}=\frac{\frac{1}{4x-5}}{1-\frac{1}{4x-5}}$$

$$=\frac{1}{4x-5}\times\frac{4x-5}{4x-5-1}$$

$=\frac{1}{4x-6}$ **(Ans.)**

Question ▸ 14 P = a + b, Q = a − b and R = 3x − 1.

[Navy Anchorage School and College, Chattogram]

a. Find the value of x when, $\log_{\sqrt{8}}x=3\frac{1}{3}$ — 2

b. If $x=\sqrt[3]{P}+\sqrt[3]{Q}$ and $\sqrt[3]{PQ}=c$, then prove that $x^3-3cx-2a=0$ — 4

c. Impose a condition on x so that the series $\frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3}+\ldots$ will have an infinite sum and find than sum. — 4

Solution to the question no. 14

a Given, $\log_{\sqrt{8}}x=3\frac{1}{3}$

Or, $\log_{\sqrt{8}}x=\frac{10}{3}$

$\therefore x=\sqrt{8}^{\frac{10}{3}}$

$=8^{\frac{1}{2}\cdot\frac{10}{3}}$

$=8^{\frac{5}{3}}$

$=2^{3\cdot\frac{5}{3}}$

$=2^5=32$

$\therefore x=32$ **(Ans.)**

b Given, P = a + b and Q = a − b

$x=\sqrt[3]{P}+\sqrt[3]{Q}$ and $\sqrt[3]{PQ}=c$

or, $\sqrt[3]{(a+b)(a-b)}=c$

or, $a^2-b^2=c^3$

Here, $x=\sqrt[3]{a+b}+\sqrt[3]{a-b}$

Or, $x=(a+b)^{\frac{1}{3}}+(a-b)^{\frac{1}{3}}$

Or, $x^3=\left\{(a+b)^{\frac{1}{3}}+(a-b)^{\frac{1}{3}}\right\}^3$ [taking cube on both sides]

Or, $x^3=\left\{(a+b)^{\frac{1}{3}}\right\}^3+\left\{(a-b)^{\frac{1}{3}}\right\}^3+3(a+b)^{\frac{1}{3}}.(a-b)^{\frac{1}{3}}.\left\{(a+b)^{\frac{1}{3}}+(a-b)^{\frac{1}{3}}\right\}$; $[\because (x+y)^3=x^3+y^3+3xy(x+y)]$

Or, $x^3=a+b+a-b+3(a^2-b^2)^{\frac{1}{3}}.x$; $\left[\because x=(a+b)^{\frac{1}{3}}+(a-b)^{\frac{1}{3}}\right]$

Or, $x^3 = 2a + 3(c^3)^{\frac{1}{3}}.x$
Or, $x^3 = 2a + 3cx$
Or, $x^3 - 3cx - 2a = 0$ **(Proved)**

c Given series,

$$\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \ldots\ldots\ldots$$

$$= \frac{1}{(3x-1)} + \frac{1}{(3x-1)^2} + \frac{1}{(3x-1)^3} + \ldots\ldots; \text{[Given, } R = 3x - 1\text{]}$$

Here, first term, $a = \frac{1}{3x-1}$

Common ratio, $r = \frac{1}{(3x-1)^2} \div \frac{1}{3x-1}$

$$= \frac{1}{(3x-1)^2} \times \frac{3x-1}{1} = \frac{1}{3x-1}$$

The series will have a sum upto infinity,
if and only if $|r| < 1$

$\therefore \frac{1}{3x-1} < 1$	Or, $\frac{1}{3x-1} > -1$
Or, $3x - 1 > 1$	Or, $3x - 1 < -1$
Or, $3x - 1 + 1 > 1 + 1$	Or, $3x < -1 + 1$
Or, $3x > 2$	Or, $3x < 0$
$\therefore x > \frac{2}{3}$	$\therefore x < 0$

$\therefore$ If $x > \frac{2}{3}$ Or, $x < 0$ then the series will have a sum up to infinity **(Ans.)**

Now, the sum of the series upto infinity,

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{3x-1}}{1 - \frac{1}{3x-1}} = \frac{\frac{1}{3x-1}}{\frac{3x-1-1}{3x-1}} = \frac{1}{3x-2} \textbf{ (Ans.)}$$

Question ▶ 15 Consider the following infinite series:

$$1 + \frac{1}{1+3x} + \frac{1}{(1+3x)^2} + \frac{1}{(1+3x)^3} + \text{-----------------}$$

[SCHOLARSHOME, Sylhet]

a. If x = 1, then find the common ratio of the given series. 2

b. Find the sum of first 7 terms of the series, when $x = \frac{1}{3}$. 4

c. Impose the condition on x, so that the series will have an infinite sum and find the sum. 4

Solution to the question no. 15

a Given the series, $1 + \frac{1}{1+3x} + \frac{1}{(1+3x)^2} + \frac{1}{(1+3x)^3} + \ldots\ldots$

If x = 1, the series, $1 + \frac{1}{1+3.1} + \frac{1}{(1+3.1)^2} + \frac{1}{(1+3.1)^3} + \ldots\ldots$

$$= 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\ldots$$

$\therefore$ Common ratio, $r = \frac{2^{nd} \text{ term}}{1^{st} \text{ term}}$

$$= \frac{\frac{1}{4}}{1}$$

$$= \frac{1}{4} \textbf{ (Ans.)}$$

b If $x = \frac{1}{3}$, the series,

$$1 + \frac{1}{1+3.\frac{1}{3}} + \frac{1}{\left(1+3.\frac{1}{3}\right)^2} + \frac{1}{\left(1+3.\frac{1}{3}\right)^3} + \ldots\ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$$

Whose 1^{st} term, a = 1

Common ratio, $r = \frac{\frac{1}{2}}{1} = \frac{1}{2} < 1$

We know, sum of 1^{st} n terms of a geometric series,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$\therefore$ Sum of 1^{st} 7 terms of the series, $S_7 = \frac{a(1-r^7)}{1-r}$

$$= \frac{1\left\{1 - \left(\frac{1}{2}\right)^7\right\}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{2^7 - 1}{2^7}}{\frac{1}{2}}$$

$$= \frac{128 - 1}{128} \times 2$$

$$= \frac{127}{64} \textbf{ (Ans.)}$$

c Here, common ratio of the series, $r = \frac{\frac{1}{1+3x}}{1}$

$$= \frac{1}{1+3x}$$

Sum to infinity of the series will exist if, $|r| < 1$

$$\text{Or, } \left|\frac{1}{1+3x}\right| < 1.$$

That is, $-1 < \frac{1}{1+3x} < 1$

$\therefore -1 < \frac{1}{1+3x}$	Or, $\frac{1}{1+3x} < 1$
Or, $-1 > 1 + 3x$	Or, $1 + 3x > 1$
Or, $-1 - 1 > 1 + 3x - 1$	Or, $1 + 3x - 1 > 1 - 1$
Or, $-2 > 3x$	Or, $3x > 0$
$\therefore x < -\frac{2}{3}$	$\therefore x > 0$

$\therefore$ Required conditions: $x > 0$ or, $x < -\frac{2}{3}$ **(Ans.)**

Now, sum to infinity of the series

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1 - \frac{1}{1+3x}}$$

$$= \frac{1}{\frac{1+3x-1}{1+3x}}$$

$$= \frac{1+3x}{3x} \textbf{ (Ans.)}$$

Question ▶ 16 $1 + \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3} + \ldots \ldots \ldots \ldots \ldots \ldots \infty$ is an infinite series. *[Jalalabad Cantonment Public School & College, Sylhet]*

a. What is the name of the series? Find its common ratio. 2
b. Impose the condition on x for the sum of infinity of the series. 4
c. Find the sum of given series. 4

Solution to the question no. 16

a The name of the given series is geometrical series.

Common ratio, $r = \dfrac{\frac{1}{(x+1)^2}}{\frac{1}{x+1}} = \frac{1}{(x+1)^2} \times \frac{(x+1)}{1}$

$= \frac{1}{x+1}$ **(Ans.)**

b Here, 1st term, a = 1

And common ratio, $r = \dfrac{\frac{1}{x+1}}{1} = \frac{1}{1+x}$

The sum of the series up to infinite will be infinite if $|r| < 1$.

Or, $-1 < r < 1$

Or, $-1 < \frac{1}{1+x} < 1$

Now, $-1 < \frac{1}{1+x}$

Or, $-1 > 1 + x$

Or, $-1 - 1 > 1 + x - 1$; [Substracting 1 from both sides]

Or, $-2 > x \therefore x < -2$

Again, $\frac{1}{1+x} < 1$

Or, $1 + x > 1$

Or, $1 + x - 1 > 1 - 1$; [Substracting 1 from both]

$\therefore x > 0$

$\therefore$ Conditions are : $x > 0$ or, $x < -2$ **(Ans.)**

c Sum upto infinity, $S_\infty = \frac{a}{1-r}$

$= \dfrac{1}{1 - \frac{1}{x+1}}$; [from a]

$= \dfrac{1}{\frac{x+1-1}{x+1}} = \frac{x+1}{x}$ **(Ans.)**

Question ▶ 17 $\frac{1}{2x-5} + \frac{1}{(2x-5)^2} + \frac{1}{(2x-5)^3} + \ldots\ldots\ldots\ldots\ldots\ldots$ is a series. *[The Sylhet Khajanchibari International School & College, Sylhet]*

a. Find the series if x = 4, what is the common ratio of the obtained series. 2
b. Find the 9th term and the sum of first 10 terms of series when x = 5. 4
c. Find the condition which should be imposed on x, so that the given series will have a sum up to infinity and find the sum. 4

Solution to the question no. 17

a Given, the series, $\frac{1}{2x-5} + \frac{1}{(2x-5)^2} + \frac{1}{(2x-5)^3} + \ldots \ldots \ldots$

If x = 4, the series,

$\frac{1}{2 \times 4 - 5} + \frac{1}{(2 \times 4 - 5)^2} + \frac{1}{(2 \times 4 - 5)^3} + \ldots \ldots$

$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots \ldots \ldots$ **(Ans.)**

$\therefore$ Common ratio, $r = \frac{1}{3^2} \div \frac{1}{3} = \frac{1}{3}$ **(Ans.)**

b If x = 5, the series,

$\frac{1}{2 \times 5 - 5} + \frac{1}{(2 \times 5 - 5)^2} + \frac{1}{(2 \times 5 - 5)^3} + \ldots \ldots$

$= \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots \ldots \ldots$

$\therefore$ First term, $a = \frac{1}{5}$

Common ratio, $r = \frac{1}{5^2} \div \frac{1}{5} = \frac{1}{5}$

We know, nth term of the geometric series $= ar^{n-1}$

$\therefore$ 9th term $= \frac{1}{5} \cdot \left(\frac{1}{5}\right)^{9-1}$

$= \frac{1}{5} \cdot \frac{1}{5^8} = \frac{1}{5^9}$ **(Ans.)**

Again, sum of first n terms, $S_n = \frac{a(1-r^n)}{1-r}$, when $r < 1$

$\therefore$ Sum of first 10 terms, $S_{10} = \dfrac{\frac{1}{5}\left(1 - \frac{1}{5^{10}}\right)}{1 - \frac{1}{5}}$

$= \dfrac{\frac{1}{5} \cdot \frac{5^{10}-1}{5^{10}}}{\frac{4}{5}}$

$= \frac{1}{5} \times \frac{5}{4} \times \frac{5^{10}-1}{5^{10}}$

$= \frac{5^{10}-1}{4 \times 5^{10}}$ **(Ans.)**

c Given, 1st term of the series, $a = \frac{1}{2x-5}$

Common ratio, $r = \frac{1}{(2x-5)^2} \div \frac{1}{(2x-5)}$

$= \frac{1}{2x-5}$

Sum to infinity of the series will exist if $|r| < 1$

That is if,

$\left|\frac{1}{2x-5}\right| < 1.$

$\therefore \frac{1}{2x-5} < 1$

Or, $2x - 5 > 1$

Or, $2x - 5 + 5 > 1 + 5$

Or, $2x > 6$

$\therefore x > 3$

Again,

$-\left(\frac{1}{2x-5}\right) < -1$

$\frac{1}{2x-5} > -1$

Or, $2x - 5 < -1$

Or, $2x - 5 + 5 < -1 + 5$

Or, $2x < 4$

$\therefore x < 2$

$\therefore$ Required condition, $x > 3$ Or, $x < 2$ **(Ans.)**

Now, sum to infinity of the series, $S_\infty = \frac{a}{1-r}$

$= \dfrac{\frac{1}{2x-5}}{1 - \frac{1}{2x-5}}$

$= \frac{1}{2x-5} \times \frac{2x-5}{2x-6}$

$= \frac{1}{2x-6}$ **(Ans.)**

Question ▸ 18 Follow the series given below: $1 + \frac{1}{x+1} + \frac{1}{(x+1)^2} + ...$ *[Jashore English School and College (JESC), Jashore]*

a. What is the name of the series and why? find its common ratio. 2

b. Impose the condition on x for the sum of infinity of the series. 4

c. Find the sum of the given series. 4

Solution to the question no. 18

a The name of the given series is geometrical series. Because, there is a common ratio of the series.

$$\text{Common ratio, } r = \frac{\frac{1}{(x+1)^2}}{\frac{1}{x+1}}$$

$$= \frac{1}{(x+1)^2} \times \frac{(x+1)}{1}$$

$$= \frac{1}{x+1} \textbf{ (Ans.)}$$

b Here, 1st term, $a = 1$

And common ratio, $r = \frac{\frac{1}{x+1}}{1} = \frac{1}{1+x}$

The sum of the series up to infinity will be infinite if $|r| < 1$.

Or, $-1 < r < 1$

Or, $-1 < \frac{1}{1+x} < 1$

Now, $-1 < \frac{1}{1+x}$

Or, $-1 > 1 + x$

Or, $-1 - 1 > 1 + x - 1$ [Substracting 1 from both sides]

Or, $-2 > x \therefore x < -2$

Again, $\frac{1}{1+x} < 1$

Or, $1 + x > 1$

Or, $1 + x - 1 > 1 - 1$ [Substracting 1 from both sides]

$\therefore x > 0$

$\therefore$ Conditions are : $x > 0$, or, $x < -2$ **(Ans.)**

c Sum up to infinity, $S_\infty = \frac{a}{1-r}$

$$= \frac{1}{1 - \frac{1}{x+1}} \text{ [from a]}$$

$$= \frac{1}{\frac{x+1-1}{x+1}}$$

$$= \frac{x+1}{x} \textbf{ (Ans.)}$$